Two-Dimensional Critical Percolation:
The Full Scaling Limit

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Abstract

We use SLE_6 paths to construct a process of continuum nonsimple loops in the plane and prove that this process coincides with the full continuum scaling limit of 2D critical site percolation on the triangular lattice – that is, the scaling limit of the set of all interfaces between different clusters. Some properties of the loop process, including conformal invariance, are also proved.

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1   Introduction and Main Results

In the theory of critical phenomena it is usually assumed that a physical system near a continuous phase transition is characterized by a single length scale (the "correlation length") in terms of which all other lengths should be measured. When combined with the experimental observation that the correlation length diverges at the phase transition, this simple but strong assumption, known as the scaling hypothesis, leads to the belief that at criticality the system has no characteristic length, and is therefore invariant under scale transformations. This suggests that all thermodynamic functions at criticality are homogeneous functions, and predicts the appearance of power laws. It also means that it should be possible to rescale a critical system appropriately and obtain a continuum

model (the "continuum scaling limit") which may have more symmetries and be easier to study than the original discrete model defined on a lattice.

Indeed, thanks to the work of Polyakov [27] and others [6, 7], it was understood by physicists since the early seventies that critical statistical mechanical models should possess continuum scaling limits with a global conformal invariance that goes beyond simple scale invariance, as long as the discrete models have "enough" rotation invariance. This property gives important information, enabling the determination of two- and three-point functions at criticality, when they are nonvanishing. Because the conformal group is in general a finite dimensional Lie group, the resulting constraints are limited in number; however, the situation becomes particularly interesting in two dimensions, since there every analytic function $\omega = f(z)$ defines a conformal transformation, at least at points where $f'(z) \neq 0$. As a consequence, the conformal group in two dimensions is infinite-dimensional.

After this observation was made, a large number of critical problems in two dimensions were analyzed using conformal methods, which were applied, among others, to Ising and Potts models, Brownian motion, Self-Avoiding Walk (SAW), percolation, and Diffusion Limited Aggregation (DLA). The large body of knowledge and techniques that resulted, starting with the work of Belavin, Polyakov and Zamolodchikov [6, 7] in the early eighties, goes under the name of Conformal Field Theory (CFT). In two dimensions, one of the main goals of CFT and its most important application to statistical mechanics is a complete classification of all universality classes via irreducible representations of the infinite-dimensional Virasoro algebra.

Partly because of the success of CFT, work in recent years on critical phenomena seemed to slow down somewhat, probably due to the feeling that most of the leading problems had been resolved. Nonetheless, however powerful and successful it may be, CFT has some limitations and leaves various open problems. First of all, the theory deals primarily with correlation functions of *local* (or quasi-local) operators, and is therefore not always the best tool to investigate other quantities. Secondly, given some critical lattice model, there is no way, within the theory itself, of deciding to which CFT it corresponds. A third limitation, of a different nature, is due to the fact that the methods of CFT, although very powerful, are generally speaking not completely rigorous from a mathematical point of view.

In a somewhat surprising twist, the most recent developments in the area of two-dimensional critical phenomena have emerged in the mathematics literature and have followed a new direction, which has provided new tools and a way of coping with at least some of the limitations of CFT. The new approach may even provide a reinterpretation of CFT, and seems to be complementary to the traditional one in the sense that questions that are difficult to pose and/or answer within CFT are easy and natural in this new approach and vice versa.

These new developments came on the heels of interesting results on the scaling limits of discrete models (see, e.g., the work of Aizenman [1, 2], Benjamini-Schramm [8], Aizenman-Burchard [3], Aizenman-Burchard-Newman-Wilson [4], Aizenman-Duplantier-Aharony [5] and Kenyon [18, 19]) but they differ greatly from those because they are based on a

radically new approach whose main tool is the Stochastic Loewner Evolution (SLE), or Schramm-Loewner Evolution, as it is also known, introduced by Schramm [32]. The new approach, which is probabilistic in nature, focuses directly on non-local structures that characterize a given system, such as cluster boundaries in Ising, Potts and percolation models, or loops in the $O(n)$ model. At criticality, these non-local objects become, in the continuum limit, random curves whose distributions can be uniquely identified thanks to their conformal invariance and a certain "Markovian" property. There is a one-parameter family of SLEs, indexed by a positive real number κ, and they appear to be the only possible candidates for the scaling limits of interfaces of two-dimensional critical systems that are believed to be conformally invariant.

In particular, substantial progress has been made in recent years, thanks to SLE, in understanding the fractal and conformally invariant nature of (the scaling limit of) large percolation clusters, which has attracted much attention and is of interest both for intrinsic reasons, given the many applications of percolation, and as a paradigm for the behavior of other systems. The work of Schramm [32] and Smirnov [36] has identified the scaling limit of a certain percolation interface with SLE_6, providing, along with the work of Lawler-Schramm-Werner [25, 26] and Smirnov-Werner [40], a confirmation of many results in the physics literature, as well as some new results.

However, SLE_6 describes a single interface, which can be obtained by imposing special boundary conditions, and is not in itself sufficient to immediately describe the scaling limit of the unconstrained model (without boundary conditions) in the whole plane. In particular, not only the nature and properties, but the very existence of the scaling limit of the collection of all interfaces remained an open question. This is true of all models, such as Ising and Potts models, that are represented in terms of clusters, and where the set of all interfaces forms a collection of loops. As already indicated by Smirnov [37], such a collection of loops should have a continuum limit, that we will call the "full" scaling limit of the model. The single interface limit is ideal for analyzing certain crossing/connectivity probabilities but not so good for others; in Section 1.1 we give a few examples showing the use of the full scaling limit to represent such probabilities. In the context of percolation, in [10] the authors used SLE_6 to construct a random process of continuous loops in the plane, which was identified with the full scaling limit of critical two-dimensional percolation, but without detailed proofs. (For a discussion of whether this full scaling limit is a "black noise," see [41]. For an analysis of random processes of loops related to SLE_κ for other values of κ, and *conjectured* to correspond to the full scaling limits of other statistical mechanics models, see [33, 35, 42].)

In this paper, we complete the analysis of [10], making rigorous the connection between the construction given there and the full scaling limit of percolation, and we prove some properties of the full scaling limit, the Continuum Nonsimple Loop process, including (one version of) conformal invariance. The present work, as well as that of Smirnov [36, 37], builds on a collection of papers, including [1–3, 5, 21, 22], which provided both inspiration and essential technical results. The proofs are based on the fact that the percolation exploration path converges in distribution to the trace of chordal SLE_6, as argued by Schramm and Smirnov [32, 36–39], and in particular on a specific version of this conver-

gence that we will call statement (S) (see Section 5). We note that no detailed proof of any version of convergence to SLE_6 has been available. Nevertheless, at the request of the editor, we do not include a detailed proof of statement (S) in the present paper, as originally planned [11], due to length considerations.

However, a detailed proof of statement (S), based on Smirnov's theorem about the convergence of crossing probabilities to Cardy's formula [36], is now the topic of a separate paper [12]. We note that statement (S) is restricted to Jordan domains while no such restriction is indicated in [36, 37].

The rest of the paper is organized as follows. In Section 1.1, we provide a quick presentation of Theorems 1, 2, and 3, which represent most of our main results, with some definitions postponed until Section 2, including that of SLE_6. Section 3 is devoted to the construction mentioned in Theorem 3 of the Continuum Nonsimple Loop process in a finite region D of the plane. In Section 4, we introduce the discrete lattice model and a discrete construction analogous to the continuum one presented in Section 3. Most of the main technical results of this paper are stated in Section 5, while Section 6 contains the proofs, using (S), of those results and the results in Section 1.1.

1.1 Main Results

At the percolation critical point, with probability one there is no infinite cluster (in two dimensions), therefore the percolation cluster boundaries form loops (see Figure 1, where site percolation on the triangular lattice $\mathcal{T}$ is depicted exploiting the duality between $\mathcal{T}$ and the hexagonal lattice $\mathcal{H}$). We will refer to the continuum scaling limit (as the mesh size δ of the rescaled hexagonal lattice $\delta\mathcal{H}$ goes to zero) of the collection of all these loops as the Continuum Nonsimple Loop process. Its existence is the content of Theorem 1 and some of its properties are described in Theorem 2 below.

The Continuum Nonsimple Loop process can be described as a "gas" of loops, or more precisely, a probability measure on countable collections of continuous, nonsimple, fractal loops in the plane. Later in this paper, we will provide precise definitions of the objects involved in the next three theorems as well as detailed proofs.

Theorem 1. *In the continuum scaling limit, the probability distribution of the collection of all boundary contours of critical site percolation on the triangular lattice converges to a probability distribution on collections of continuous, nonsimple loops.*

Theorem 2. *The Continuum Nonsimple Loop process whose distribution is specified in Theorem 1 has the following properties, which are valid with probability one:*

1. *It is a random collection of countably many noncrossing continuous loops in the plane. The loops can and do touch themselves and each other many times, but there are no triple points; i.e. no three or more loops can come together at the same point, and a single loop cannot touch the same point more than twice, nor can a loop touch a point where another loop touches itself.*
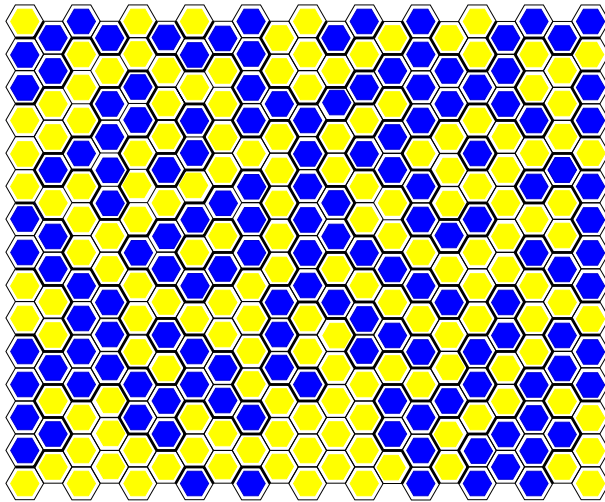
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Figure 1: Finite portion of a (site) percolation configuration on the triangular lattice $\mathcal{T}$. Each hexagon of the hexagonal lattice $\mathcal{H}$ represents a site of $\mathcal{T}$ and is assigned one of two colors. In the critical percolation model, colors are assigned randomly with equal probability. The cluster boundaries are indicated by heavy lines; some small loops appear, while other boundaries extend beyond the finite window.

2. *Any deterministic point z in the plane (i.e., chosen independently of the loop process) is surrounded by an infinite family of nested loops with diameters going to both zero and infinity; any annulus about that point with inner radius $r_1 > 0$ and outer radius $r_2 < \infty$ contains only a finite number $N(z, r_1, r_2)$ of those loops. Consequently, any two distinct deterministic points of the plane are separated by loops winding around each of them.*

3. *Any two loops are connected by a finite "path" of touching loops.*

The next theorem makes explicit the relation between the percolation full scaling limit and SLE_6. Its proof (see Section 6) relies on an inductive procedure that makes use of SLE_6 at each step and allows to obtain collections of loops with the correct distribution.

Theorem 3. *A Continuum Nonsimple Loop process with the same distribution as in Theorem 1 can be constructed by a procedure in which each loop is obtained as the concatenation of an SLE_6 path with (a portion of) another SLE_6 path (see Figure 4). This procedure is carried out first in a finite disk $\mathbb{D}_R$ of radius R in the plane (see Section 3.2), and then an infinite volume limit, $\mathbb{D}_R \to \mathbb{R}^2$, is taken.*

Remark 1.1. *There are various possible ways to formulate the conformal invariance properties of the Continuum Nonsimple Loop process. One version is given in Theorem 7.*

Next we give some examples showing how the scaling limit of various connectivity/crossing probabilities can be expressed in terms of the loop process. Although we

cannot say whether this fact may eventually lead to exact expressions going beyond Cardy's formula, it at least shows that scaling limits of such probabilities exist and are conformally invariant (early discussions of scaling limits of connectivity functions and of the consequences of conformal invariance for such quantities are given in [1, 2]). The examples will also highlight the natural nested structure of the collection of percolation cluster boundaries in the scaling limit.

Consider first an annulus centered at z with inner radius r_1 and outer radius r_2 (see Figure 2). The scaling limit $p(r_1, r_2)$ of the probability of a crossing of the annulus (by crossing here we refer to a "monochromatic" crossing, i.e., a crossing by either of the two colors, as discussed in Section 4 – see also Figure 1) can be expressed as follows in terms of the random variable $N(z, r_1, r_2)$ defined in Theorem 2 above: $p(r_1, r_2)$ is the probability that $N(z, r_1, r_2)$ equals zero. More generally, $N(z, r_1, r_2)$ represents the scaling limit of the minimal number of cluster boundaries traversed by any path connecting the inner and outer circles of the annulus.
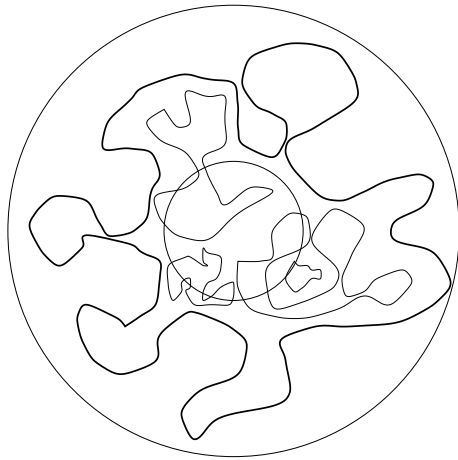

Figure 2: An annulus whose inner disc is surrounded by a continuum nonsimple loop. There is no monochromatic crossing between the inner and outer discs. Other continuum nonsimple loops are shown in the figure, but they do not affect the connectivity between the inner and outer discs.

An example with more geometric structure involves two disjoint discs D_1 and D_2 in the plane and the scaling limit $p(D_1, D_2)$ of the probability that there is a crossing from D_1 to D_2 (see Figure 3). Here we let N_1 denote the number of distinct loops in the plane that contain D_1 in their interior and D_2 in their exterior, and define N_2 in the complementary way. Then $p(D_1, D_2)$ is the probability that $N_1 = N_2 = 0$, and the scaling limit of the minimal number of cluster boundaries that must be crossed to connect D_1 to D_2 is $N_1 + N_2$.

One can also consider, as in [1, 2], the probability of a single monochromatic cluster in the exterior $\mathcal{E}$ of the union of m disjoint discs (or other regions) connecting all m disc boundaries. In the scaling limit, this can be expressed as the probability of the event that
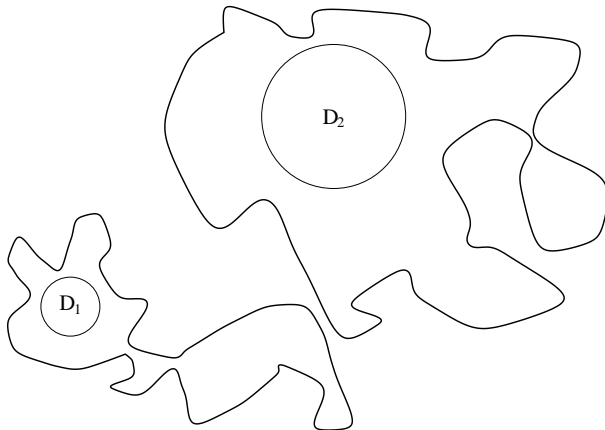
Figure 3: Each one of the two disjoint discs in the figure is surrounded by a continuum nonsimple loop that has the other disc in its exterior. The minimal number of cluster boundaries that must be crossed to connect the two discs is two.

there is a single continuous (nonsimple) curve in $\mathcal{E}$ touching all m disc boundaries that *does not cross any of the loops of the Continuum Nonsimple Loop process.*

We conclude this section by remarking that the Continuum Nonsimple Loop process is just one example of a family of "conformal loop ensembles" that are related to SLE and to the Gaussian Free Field (see [33, 35, 42, 44, 45]), and are conjectured to describe the full scaling limit of statistical mechanics models such as percolation, Ising and Potts models. The work of Lawler, Schramm, Sheffield and Werner has provided tools to define such loop ensembles in the continuum and to study some of their properties. But the key question concerning the proof of their connection to the discrete models via a continuum scaling limit remains an important open challenge, the only exception currently being percolation, as we show in this paper.

2 Preliminary Definitions and Results

We will find it convenient to identify the real plane $\mathbb{R}^2$ and the complex plane $\mathbb{C}$. We will also refer to the Riemann sphere $\mathbb{C} \cup \infty$ and the open upper half-plane $\mathbb{H} = \{x + iy : y > 0\}$ (and its closure $\overline{\mathbb{H}}$), where chordal SLE will be defined (see Section 2.3). $\mathbb{D}$ will denote the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

A domain D of the complex plane $\mathbb{C}$ is a nonempty, connected, open subset of $\mathbb{C}$; a simply connected domain D is said to be a Jordan domain if its (topological) boundary ∂D is a Jordan curve (i.e., a simple continuous loop).

We will make repeated use of Riemann's mapping theorem, which states that if D is any simply connected domain other than the entire plane $\mathbb{C}$ and $z_0 \in D$, then there is a unique conformal map f of D onto $\mathbb{D}$ such that $f(z_0) = 0$ and $f'(z_0) > 0$.

2.1 Compactification of $\mathbb{R}^2$

When taking the scaling limit as the lattice mesh size $\delta \to 0$ one can focus on fixed finite regions, $\Lambda \subset \mathbb{R}^2$, or consider the whole $\mathbb{R}^2$ at once. The second option avoids dealing with boundary conditions, but requires an appropriate choice of metric.

A convenient way of dealing with the whole $\mathbb{R}^2$ is to replace the Euclidean metric with a distance function $\Delta(\cdot,\cdot)$ defined on $\mathbb{R}^2 \times \mathbb{R}^2$ by

$$\Delta(u,v) = \inf_{\varphi} \int (1 + |\varphi|^2)^{-1} \, ds, \tag{1}$$

where the infimum is over all smooth curves $\varphi(s)$ joining u with v, parameterized by arclength s, and where $|\cdot|$ denotes the Euclidean norm. This metric is equivalent to the Euclidean metric in bounded regions, but it has the advantage of making $\mathbb{R}^2$ precompact. Adding a single point at infinity yields the compact space $\dot{\mathbb{R}}^2$ which is isometric, via stereographic projection, to the two-dimensional sphere.

2.2 The Space of Curves

In dealing with the scaling limit we use the approach of Aizenman-Burchard [3]. Denote by $\mathcal{S}_R$ the complete separable metric space of continuous curves in the closure $\overline{\mathbb{D}}_R$ of the disc $\mathbb{D}_R$ of radius R with the metric (2) defined below. Curves are regarded as equivalence classes of continuous functions from the unit interval to $\overline{\mathbb{D}}_R$, modulo monotonic reparametrizations. γ will represent a particular curve and $\gamma(t)$ a parametrization of γ; $\mathcal{F}$ will represent a set of curves (more precisely, a closed subset of $\mathcal{S}_R$). $\mathrm{d}(\cdot,\cdot)$ will denote the uniform metric on curves, defined by

$$\mathrm{d}(\gamma_1,\gamma_2) \equiv \inf \sup_{t \in [0,1]} |\gamma_1(t) - \gamma_2(t)|, \tag{2}$$

where the infimum is over all choices of parametrizations of γ_1 and γ_2 from the interval $[0,1]$. The distance between two closed sets of curves is defined by the induced Hausdorff metric as follows:

$$\mathrm{dist}(\mathcal{F},\mathcal{F}') \le \varepsilon \Leftrightarrow (\forall \, \gamma \in \mathcal{F}, \, \exists \, \gamma' \in \mathcal{F}' \text{ with } \mathrm{d}(\gamma,\gamma') \le \varepsilon, \text{ and vice versa}). \tag{3}$$

The space Ω_R of closed subsets of $\mathcal{S}_R$ (i.e., collections of curves in $\overline{\mathbb{D}}_R$) with the metric (3) is also a complete separable metric space. We denote by $\mathcal{B}_R$ its Borel σ-algebra.

For each fixed $\delta > 0$, the random curves that we consider are polygonal paths on the edges of the hexagonal lattice $\delta\mathcal{H}$, dual to the triangular lattice $\delta\mathcal{T}$. A superscript δ is added to indicate that the curves correspond to a model with a "short distance cutoff" of magnitude δ.

We will also consider the complete separable metric space $\mathcal{S}$ of continuous curves in $\dot{\mathbb{R}}^2$ with the distance

$$\mathrm{D}(\gamma_1,\gamma_2) \equiv \inf \sup_{t \in [0,1]} \Delta(\gamma_1(t),\gamma_2(t)), \tag{4}$$

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where the infimum is again over all choices of parametrizations of γ_1 and γ_2 from the interval $[0, 1]$. The distance between two closed sets of curves is again defined by the induced Hausdorff metric as follows:

$$\text{Dist}(\mathcal{F}, \mathcal{F}') \leq \varepsilon \Leftrightarrow (\forall \, \gamma \in \mathcal{F}, \, \exists \, \gamma' \in \mathcal{F}' \text{ with } \text{D}(\gamma, \gamma') \leq \varepsilon \text{ and vice versa}). \qquad (5)$$

The space Ω of closed sets of $\mathcal{S}$ (i.e., collections of curves in $\dot{\mathbb{R}}^2$) with the metric (5) is also a complete separable metric space. We denote by $\mathcal{B}$ its Borel σ-algebra.

When we talk about convergence in distribution of random curves, we always mean with respect to the uniform metric (2), while when we deal with closed collections of curves, we always refer to the metric (3) or (5).

Remark 2.1. *In this paper, the space Ω of closed sets of $\mathcal{S}$ is used for collections of exploration paths (see Section 4.1) and cluster boundary loops and their scaling limits, SLE_6 paths and continuum nonsimple loops.*

2.3 Chordal SLE in the Upper Half-Plane

The Stochastic Loewner Evolution (SLE) was introduced by Schramm [32] as a tool for studying the scaling limit of two-dimensional discrete (defined on a lattice) probabilistic models whose scaling limits are expected to be conformally invariant. In this section we define the chordal version of SLE; for more on the subject, the interested reader can consult the original paper [32] as well as the fine reviews by Lawler [23], Kager and Nienhuis [17], and Werner [43], and Lawler's book [24].

Let $\mathbb{H}$ denote the upper half-plane. For a given continuous real function U_t with $U_0 = 0$, define, for each $z \in \overline{\mathbb{H}}$, the function $g_t(z)$ as the solution to the ODE

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \qquad (6)$$

with $g_0(z) = z$. This is well defined as long as $g_t(z) - U_t \neq 0$, i.e., for all $t < T(z)$, where

$$T(z) \equiv \sup\{t \geq 0 : \min_{s \in [0, t]} |g_s(z) - U_s| > 0\}. \qquad (7)$$

Let $K_t \equiv \{z \in \overline{\mathbb{H}} : T(z) \leq t\}$ and let $\mathbb{H}_t$ be the unbounded component of $\mathbb{H} \setminus K_t$; it can be shown that K_t is bounded and that g_t is a conformal map from $\mathbb{H}_t$ onto $\mathbb{H}$. For each t, it is possible to write $g_t(z)$ as

$$g_t(z) = z + \frac{2t}{z} + o(\frac{1}{z}), \qquad (8)$$

when $z \to \infty$. The family $(K_t, t \geq 0)$ is called the **Loewner chain** associated to the driving function $(U_t, t \geq 0)$.

Definition 2.1. Chordal SLE_κ *is the Loewner chain $(K_t, t \geq 0)$ that is obtained when the driving function $U_t = \sqrt{\kappa} B_t$ is $\sqrt{\kappa}$ times a standard real-valued Brownian motion $(B_t, t \geq 0)$ with $B_0 = 0$.*

For all $\kappa \geq 0$, chordal SLE_κ is almost surely generated by a continuous random curve γ in the sense that, for all $t \geq 0$, $\mathbb{H}_t \equiv \mathbb{H} \setminus K_t$ is the unbounded connected component of $\mathbb{H} \setminus \gamma[0, t]$; γ is called the **trace** of chordal SLE_κ.

2.4   Chordal SLE in a Jordan Domain

Let $D \subset \mathbb{C}$ be a Jordan domain. By Riemann's mapping theorem, there are (many) conformal maps from the upper half-plane $\mathbb{H}$ onto D. In particular, given two distinct points $a, b \in \partial D$, there exists a conformal map f from $\mathbb{H}$ onto D such that $f(0) = a$ and $f(\infty) \equiv \lim_{|z| \to \infty} f(z) = b$. In fact, the choice of the points a and b on the boundary of D only characterizes $f(\cdot)$ up to a multiplicative factor, since $f(\lambda \cdot)$ would also do.

Suppose that $(K_t, t \geq 0)$ is a chordal SLE_κ in $\mathbb{H}$ as defined above; we define chordal SLE_κ $(\tilde{K}_t, t \geq 0)$ in D from a to b as the image of the Loewner chain $(K_t, t \geq 0)$ under f. It is possible to show, using scaling properties of SLE_κ, that the law of $(\tilde{K}_t, t \geq 0)$ is unchanged, up to a linear time-change, if we replace $f(\cdot)$ by $f(\lambda \cdot)$. This makes it natural to consider $(\tilde{K}_t, t \geq 0)$ as a process from a to b in D, ignoring the role of f.

We are interested in the case $\kappa = 6$, for which $(K_t, t \geq 0)$ is generated by a continuous, nonsimple, non-self-crossing curve γ with Hausdorff dimension $7/4$. We will denote by $\gamma_{D,a,b}$ the image of γ under f and call it the trace of chordal SLE_6 in D from a to b; $\gamma_{D,a,b}$ is a continuous nonsimple curve inside D from a to b, and it can be given a parametrization $\gamma_{D,a,b}(t)$ such that $\gamma_{D,a,b}(0) = a$ and $\gamma_{D,a,b}(1) = b$, so that we are in the metric framework described in Section 2.2. It will be convenient to think of $\gamma_{D,a,b}$ as an oriented path, with orientation from a to b.

2.5   Radó's Theorem

We present here Radó's theorem [29] (see also Theorem 2.11 of [28]), which deals with sequences of Jordan domains and the corresponding conformal maps from the unit disc, and will be used in the proof of the key Lemma 5.3.

Since the theorem deals with Jordan domains, the conformal maps from the unit disc to those domains have a continuous extension to $\mathbb{D} \cup \partial \mathbb{D}$. With a slight abuse of notation, we do not distinguish between the conformal maps and their continuous extensions.

Theorem 4. *For $k = 1, 2, \ldots$, let J_k and J be Jordan curves parameterized respectively by $\phi_k(t)$ and $\phi(t)$, $t \in [0, 1]$, and let f_k and f be conformal maps from $\mathbb{D}$ onto the inner domains of J_k and J such that $f_k(0) = f(0)$ and $f_k'(0) > 0$, $f'(0) > 0$ for all k. If $\phi_k \to \phi$ as $k \to \infty$ uniformly in $[0, 1]$ then $f_k \to f$ as $k \to \infty$ uniformly in $\overline{\mathbb{D}}$.*

The type of convergence of sequences of Jordan domains $\{D_k\}$ to a Jordan domain D that will be encountered in Lemma 5.3 is such that ∂D_k converges, as $k \to \infty$, to ∂D in the uniform metric (2) on continuous curves, which is clearly sufficient to apply Theorem 4.

3   Construction of the Continuum Nonsimple Loops

3.1   Construction of a Single Loop

As a preview to the full construction, we explain how to construct a single loop using two SLE_6 paths inside a domain D whose boundary is assumed to have a given orientation (clockwise or counterclockwise). This is done in three steps (see Figure 4), of which the first consists in choosing two points a and b on the boundary ∂D of D and "running" a chordal SLE_6, $\gamma = \gamma_{D,a,b}$, from a to b inside D. As explained in Section 2.4, we consider γ as an oriented path, with orientation from a to b. The set $D \setminus \gamma_{D,a,b}[0,1]$ is a countable union of its connected components, which are open and simply connected. If z is a deterministic point in D, then with probability one, z is not touched by γ [30] and so it belongs to a unique domain in $D \setminus \gamma_{D,a,b}[0,1]$ that we denote $D_{a,b}(z)$.
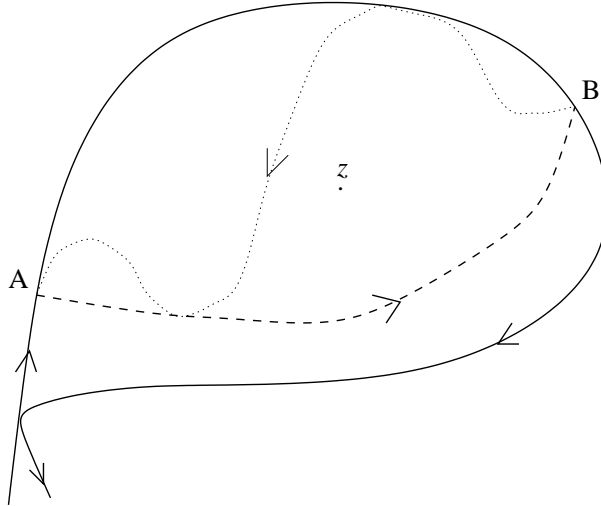

Figure 4: Construction of a continuum loop around z in three steps. A domain D is formed by the solid curve. The dashed curve is an excursion $\mathcal{E}$ (from A to B) of an SLE_6 in D that creates a subdomain D' containing z. The dotted curve γ' is an SLE_6 in D' from B to A. A loop is formed by $\mathcal{E}$ followed by γ'.

The elements of $D \setminus \gamma_{D,a,b}[0,1]$ can be conveniently characterized in terms of how a point z in the interior of the component was first "trapped" at some time t_1 by $\gamma[0,t_1]$, perhaps together with either $\partial_{a,b}D$ or $\partial_{b,a}D$ (the portions of the boundary ∂D from a to b counterclockwise or clockwise respectively) — see Figure 5: (1) those components whose boundary contains a segment of $\partial_{b,a}D$ between two successive visits at $\gamma_0(z) = \gamma(t_0)$ and $\gamma_1(z) = \gamma(t_1)$ to $\partial_{b,a}D$ (where here and below $t_0 < t_1$), (2) the analogous components with $\partial_{b,a}D$ replaced by the other part of the boundary $\partial_{a,b}D$, (3) those components formed when $\gamma_0(z) = \gamma(t_0) = \gamma(t_1) = \gamma_1(z) \in D$ with γ winding about z in a counterclockwise direction between t_0 and t_1, and finally (4) the analogous clockwise components.
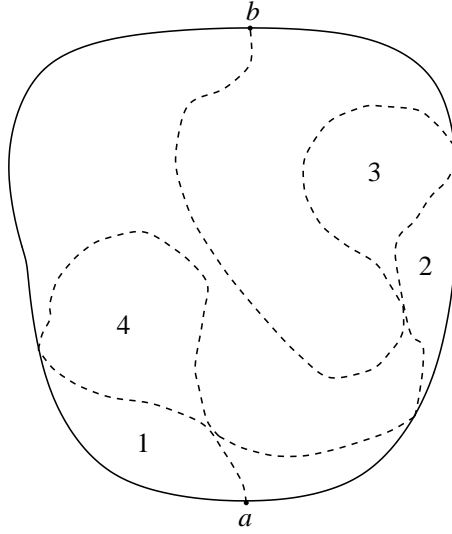
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Figure 5: Schematic diagram showing the four types of (sub)domains formed by a dashed curve γ from a to b inside a domain whose boundary is the solid curve.

We give to the boundary of a domain of type 3 or 4 the orientation induced by how the curve γ winds around the points inside that domain. For a domain $D' \ni z$ of type 1 or 2 which is produced by an "excursion" $\mathcal{E}$ from $\gamma_0(z) \in \partial D$ to $\gamma_1(z) \in \partial D$, the part of the boundary that corresponds to the inner perimeter of the excursion $\mathcal{E}$ (i.e., the perimeter of γ seen from z) is oriented according to the direction of γ, i.e., from $\gamma_0(z)$ to $\gamma_1(z)$.

If we assume that ∂D is oriented from a to b clockwise, then the boundaries of domains of type 2 have a well defined orientation, while the boundaries of domains of type 1 do not, since they are composed of two parts which are both oriented from the beginning to the end of the excursion that produced the domain.

Now, let D' be a domain of type 1 and let A and B be respectively the starting and ending point of the excursion that generated D'. The second step to construct a loop is to run a chordal SLE_6, $\gamma' = \gamma_{D',B,A}$, inside D' from B to A; the third and final step consists in pasting together $\mathcal{E}$ and γ'.

Running γ' inside D' from B to A partitions $D' \backslash \gamma'$ into new domains. Notice that if we assign an orientation to the boundaries of these domains according to the same rules used above, all of those boundaries have a well defined orientation, so that the construction of loops just presented can be iterated inside each one of these domains (as well as inside each of the domains of type 2, 3 and 4 generated by $\gamma_{D,a,b}$ in the first step). This will be done in the next section.

3.2   The Full Construction Inside The Unit Disc

In this section we define the Continuum Nonsimple Loop process inside the unit disc $\mathbb{D} = \mathbb{D}_1$ via an inductive procedure. Later, in order to define the continuum nonsimple

loops in the whole plane, the unit disc will be replaced by a growing sequence of large discs, $\mathbb{D}_R$, with $R \to \infty$ (see Theorem 6). The basic ingredient in the algorithmic construction, given in the previous section, consists of a chordal SLE_6 path $\gamma_{D,a,b}$ between two points a and b of the boundary ∂D of a given simply connected domain $D \subset \mathbb{C}$.

We will organize the inductive procedure in steps, each one corresponding to one SLE_6 inside a certain domain generated by the previous steps. To do that, we need to order the domains present at the end of each step, so as to choose the one to use in the next step. For this purpose, we introduce a deterministic countable set of points $\mathcal{P}$ that are dense in $\mathbb{C}$ and are endowed with a deterministic order (here and below by deterministic we mean that they are assigned before the beginning of the construction and are independent of the SLE_6's).

The first step consists of an SLE_6 path, $\gamma_1 = \gamma_{\mathbb{D},-i,i}$, inside $\mathbb{D}$ from $-i$ to i, which produces many domains that are the connected components of the set $\mathbb{D} \setminus \gamma_1[0,1]$. These domains can be priority-ordered according to the maximal x- or y- coordinate distances between points on their boundaries and using the rank of the points in $\mathcal{P}$ (contained in the domains) to break ties, as follows. For a domain D, let $\mathrm{d}_m(D)$ be the maximal x- or y-distance between points on its boundary, whichever is greater. Domains with larger d_m have higher priority, and if two domains have the same d_m, the one containing the highest ranking point of $\mathcal{P}$ from those two domains has higher priority. The priority order of domains of course changes as the construction proceeds and new domains are formed.

The second step of the construction consists of an SLE_6 path, γ_2, that is produced in the domain with highest priority (after the first step). Since all the domains that are produced in the construction are Jordan domains, as explained in the discussion following Corollary 5.1, for all steps we can use the definition of chordal SLE given in Section 2.4.

As a result of the construction, the SLE_6 paths are naturally ordered: $\{\gamma_j\}_{j \in \mathbb{N}}$. It will be shown (see especially the proof of Theorem 5 below) that every domain that is formed during the construction is eventually used (this is in fact one important requirement in deciding how to order the domains and therefore how to organize the construction).

So far we have not explained how to choose the starting and ending points of the SLE_6 paths on the boundaries of the domains. In order to do this, we give an orientation to the boundaries of the domains produced by the construction according to the rules explained in Section 3.1. We call **monochromatic** a boundary which gets, as a consequence of those rules, a well defined (clockwise or counterclockwise) orientation; the choice of this term will be clarified when we discuss the lattice version of the loop construction below. We will generally take our initial domain $\mathbb{D}_1$ (or $\mathbb{D}_R$) to have a monochromatic boundary (either clockwise or counterclockwise orientation).

It is easy to see by induction that the boundaries that are not monochromatic are composed of two "pieces" joined at two special points (call them A and B, as in the example of Section 3.1), such that one piece is a portion of the boundary of a previous domain, and the other is the inner perimeter of an excursion (see again Section 3.1). Both pieces are oriented in the same direction, say from A to B (see Figure 4).

For a domain whose boundary is not monochromatic, we make the "natural" choice of starting and ending points, corresponding to the end and beginning of the excursion

that produced the domain (the points B and A respectively, in the example above). As explained in Section 3.1, when such a domain is used with this choice of points on the boundary, a loop is produced, together with other domains, whose boundaries are all monochromatic.

For a domain whose boundary is monochromatic, and therefore has a well defined orientation, there are various procedures which would yield the "correct" distribution for the resulting Continuum Nonsimple Loop process; one possibility is as follows.

Given a domain D, a and b are chosen so that, of all pairs (u, v) of points in ∂D, they maximize $|\mathrm{Re}(u - v)|$ if $|\mathrm{Re}(u - v)| \geq |\mathrm{Im}(u - v)|$, or else they maximize $|\mathrm{Im}(u - v)|$. If the choice is not unique, to restrict the number of pairs one looks at those pairs, among the ones already obtained, that maximize the other of $\{|\mathrm{Re}(u - v)|, |\mathrm{Im}(u - v)|\}$. Notice that this leaves at most two pairs of points; if that's the case, the pair that contains the point with minimal real (and, if necessary, imaginary) part is chosen. The iterative procedure produces a loop every time a domain whose boundary is not monochromatic is used. Our basic loop process consists of the collection of all loops generated by this inductive procedure (i.e., the limiting object obtained from the construction by letting the number of steps $k \to \infty$), to which we add a "trivial" loop for each z in $\mathbb{D}$, so that the collection of loops is closed in the appropriate sense [3]. The Continuum Nonsimple Loop process in the whole plane is introduced in Theorem 6, Section 5. There, a "trivial" loop for each $z \in \mathbb{C} \cup \infty$ has to be added to make the space of loops closed.

4 Lattices and Paths

We will denote by $\mathcal{T}$ the two-dimensional triangular lattice, whose sites we think of as the elementary cells of a regular hexagonal lattice $\mathcal{H}$ embedded in the plane as in Figure 1. Two hexagons are **neighbors** if they are adjacent, i.e., if they have a common edge. A sequence $(\xi_0, \ldots, \xi_n)$ of hexagons such that ξ_{i-1} and ξ_i are neighbors for all $i = 1, \ldots, n$ and $\xi_i \neq \xi_j$ whenever $i \neq j$ will be called a $\mathcal{T}$**-path** and denoted by π. If the first and last sites of the path are neighbors, the path will be called a $\mathcal{T}$**-loop**.

A finite set D of hexagons is **connected** if any two hexagons in D can be joined by a $\mathcal{T}$-path contained in D. We say that a finite set D of hexagons is **simply connected** if both D and its complement are connected. For a simply connected set D of hexagons, we denote by ΔD its **external site boundary**, or **s-boundary** (i.e., the set of hexagons that do not belong to D but are adjacent to hexagons in D), and by ∂D the topological boundary of D when D is considered as a domain of $\mathbb{C}$. We will call a bounded, simply connected subset D of $\mathcal{T}$ a **Jordan set** if its s-boundary ΔD is a $\mathcal{T}$-loop.

For a Jordan set $D \subset \mathcal{T}$, a vertex $x \in \mathcal{H}$ that belongs to ∂D can be either of two types, according to whether the edge incident on x that is not in ∂D belongs to a hexagon in D or not. We call a vertex of the second type an **e-vertex** (e for "external" or "exposed").

Given a Jordan set D and two e-vertices x, y in ∂D, we denote by $\partial_{x,y}D$ the portion of ∂D traversed counterclockwise from x to y, and call it the **right boundary**; the remaining part of the boundary is denote by $\partial_{y,x}D$ and is called the **left boundary**. Analogously,

the portion of $\Delta_{x,y}D$ of ΔD whose hexagons are adjacent to $\partial_{x,y}D$ is called the **right s-boundary** and the remaining part the **left s-boundary**.

A **percolation configuration** $\sigma = \{\sigma(\xi)\}_{\xi \in \mathcal{T}} \in \{-1, +1\}^{\mathcal{T}}$ on $\mathcal{T}$ is an assignment of -1 (equivalently, yellow) or $+1$ (blue) to each site of $\mathcal{T}$ (i.e., to each hexagon of $\mathcal{H}$ – see Figure 1). For a domain D of the plane, the restriction to the subset $D \cap \mathcal{T}$ of $\mathcal{T}$ of the percolation configuration σ is denoted by σ_D. On the space of configurations $\Sigma = \{-1, +1\}^{\mathcal{T}}$, we consider the usual product topology and denote by $\mathbb{P}$ the uniform measure, corresponding to Bernoulli percolation with equal density of yellow (minus) and blue (plus) hexagons, which is critical percolation in the case of the triangular lattice.

A (percolation) **cluster** is a maximal, connected, monochromatic subset of $\mathcal{T}$; we will distinguish between blue (plus) and yellow (minus) clusters. The **boundary** of a cluster D is the set of edges of $\mathcal{H}$ that surround the cluster (i.e., its Peierls contour); it coincides with the topological boundary of D considered as a domain of $\mathbb{C}$. The set of all boundaries is a collection of "nested" simple loops along the edges of $\mathcal{H}$.

Given a percolation configuration σ, we associate an arrow to each edge of $\mathcal{H}$ belonging to the boundary of a cluster in such a way that the hexagon to the right of the edge with respect to the direction of the arrow is blue (plus). The set of all boundaries then becomes a collection of nested, oriented, simple loops. A **boundary path** (or **b-path**) γ is a sequence $(e_0, \dots, e_n)$ of distinct edges of $\mathcal{H}$ belonging to the boundary of a cluster and such that e_{i-1} and e_i meet at a vertex of $\mathcal{H}$ for all $i = 1, \dots, n$. To each b-path, we can associate a direction according to the direction of the edges in the path.

Given a b-path γ, we denote by $\Gamma_B(\gamma)$ (respectively, $\Gamma_Y(\gamma)$) the set of blue (resp., yellow) hexagons (i.e., sites of $\mathcal{T}$) adjacent to γ; we also let $\Gamma(\gamma) \equiv \Gamma_B(\gamma) \cup \Gamma_Y(\gamma)$.

4.1 The Percolation Exploration Process and Path

For a Jordan set $D \subset \mathcal{T}$ and two e-vertices x, y in ∂D, imagine coloring blue all the hexagons in $\Delta_{x,y}D$ and yellow all those in $\Delta_{y,x}D$. Then, for any percolation configuration σ_D inside D, there is a unique b-path γ from x to y which separates the blue cluster adjacent to $\Delta_{x,y}D$ from the yellow cluster adjacent to $\Delta_{y,x}D$. We call $\gamma = \gamma_{D,x,y}(\sigma_D)$ a **percolation exploration path** (see Figure 6).

An exploration path γ can be decomposed into **left excursions** $\mathcal{E}$, i.e., maximal b-subpaths of γ that do not use edges of the left boundary $\partial_{y,x}D$. Successive left excursions are separated by portions of γ that contain only edges of the left boundary $\partial_{y,x}D$. Analogously, γ can be decomposed into **right excursions**, i.e., maximal b-subpaths of γ that do not use edges of the right boundary $\partial_{x,y}D$. Successive right excursions are separated by portions of γ that contain only edges of the right boundary $\partial_{x,y}D$.

Notice that the exploration path $\gamma = \gamma_{D,x,y}(\sigma_D)$ only depends on the percolation configuration σ_D inside D and the positions of the e-vertices x and y; in particular, it does not depend on the color of the hexagons in ΔD, since it is defined by imposing fictitious $\pm$ boundary conditions on D. To see this more clearly, we next show how to construct the percolation exploration path dynamically, via the **percolation exploration process** defined below.

Given a Jordan set $D \subset \mathcal{T}$ and two e-vertices x, y in ∂D, assign to $\partial_{x,y} D$ a counterclockwise orientation (i.e., from x to y) and to $\partial_{y,x} D$ a clockwise orientation. Call e_x the edge incident on x that does not belong to ∂D and orient it in the direction of x; this is the "starting edge" of an exploration procedure that will produce an oriented path inside D along the edges of $\mathcal{H}$, together with two *nonsimple* monochromatic paths on $\mathcal{T}$. ¿From e_x, the process moves along the edges of hexagons in D according to the rules below. At each step there are two possible edges (left or right edge with respect to the current direction of exploration) to choose from, both belonging to the same hexagon ξ contained in D or ΔD.

- If ξ belongs to D and has not been previously "explored," its color is determined by flipping a fair coin and then the edge to the left (with respect to the direction in which the exploration is moving) is chosen if ξ is blue (plus), or the edge to the right is chosen if ξ is yellow (minus).

- If ξ belongs to D and has been previously explored, the color already assigned to it is used to choose an edge according to the rule above.

- If ξ belongs to the right external boundary $\Delta_{x,y} D$, the left edge is chosen.

- If ξ belongs to the left external boundary $\Delta_{y,x} D$, the right edge is chosen.

- The exploration process stops when it reaches b.

We can assign an arrow to each edge in the path in such a way that the hexagon to the right of the edge with respect to the arrow is blue; for edges in ∂D, we assign the arrows according to the direction assigned to the boundary. In this way, we get an oriented path, whose shape and orientation depend solely on the color of the hexagons explored during the construction of the path.

When we present the discrete construction, we will encounter Jordan sets D with two e-vertices $x, y \in \partial D$ assigned in some way to be discussed later. Such domains will have either monochromatic (plus or minus) boundaries or $\pm$ boundary conditions, corresponding to having both $\Delta_{x,y} D$ and $\Delta_{y,x} D$ monochromatic, but of different colors.

As explained, the exploration path $\gamma_{D,x,y}$ does not depend on the color of ΔD, but the interpretation of $\gamma_{D,x,y}$ does. For domains with $\pm$ boundary conditions, the exploration path represents the interface between the yellow cluster containing the yellow portion of the s-boundary of D and the blue cluster containing its blue portion.

For domains with monochromatic blue (resp., yellow) boundary conditions, the exploration path represents portions of the boundaries of yellow (resp., blue) clusters touching $\partial_{y,x} D$ and adjacent to blue (resp., yellow) hexagons that are the starting point of a blue (resp., yellow) path (possibly an empty path) that reaches $\partial_{x,y} D$, pasted together using portions of $\partial_{y,x} D$.

In order to study the continuum scaling limit of an exploration path, we introduce the following definitions.
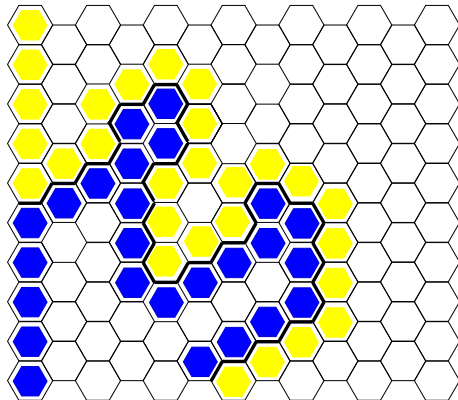
Figure 6: Percolation exploration process in a portion of the hexagonal lattice with ± boundary conditions on the first column, corresponding to the boundary of the region where the exploration is carried out. The colored hexagons that do not belong to the first column have been "explored" during the exploration process. The heavy line between yellow (light) and blue (dark) hexagons is the exploration path produced by the exploration process.

Definition 4.1. *Given a Jordan domain D of the plane, we denote by D^δ the largest Jordan set of hexagons of the scaled hexagonal lattice $\delta\mathcal{H}$ that is contained in D, and call it the δ-**approximation** of D.*

It is clear that ∂D^δ converges to ∂D in the metric (2).

Definition 4.2. *Let D be a Jordan domain of the plane and D^δ its δ-approximation. For $a, b \in \partial D$, choose the pair (x_a, x_b) of e-vertices in ∂D^δ closest to, respectively, a and b (if there are two such vertices closest to a, we choose, say, the first one encountered going clockwise along ∂D^δ, and analogously for b). Given a percolation configuration σ, we define the **exploration path** $\gamma^\delta_{D,a,b}(\sigma) \equiv \gamma_{D^\delta, x_a, x_b}(\sigma)$.*

For a fixed $\delta > 0$, the measure $\mathbb{P}$ on percolation configurations σ induces a measure $\mu^\delta_{D,a,b}$ on exploration paths $\gamma^\delta_{D,a,b}(\sigma)$. In the continuum scaling limit, $\delta \to 0$, one is interested in the weak convergence of $\mu^\delta_{D,a,b}$ to a measure $\mu_{D,a,b}$ supported on continuous curves, with respect to the uniform metric (2) on continuous curves.

One of the main tools in this paper is the result on convergence to SLE_6 announced by Smirnov [36] (see also [37]), whose detailed proof is to appear [38]: *The distribution of $\gamma^\delta_{D,a,b}$ converges, as $\delta \to 0$, to that of the trace of chordal SLE_6 inside D from a to b, with respect to the uniform metric (2) on continuous curves.*

Actually, we will rather use a slightly stronger conclusion, given as statement (S) at the beginning of Section 5 below, a version of which, according to [40] (see p. 734 there), and [39], will be contained in [38]. This stronger statement is that the convergence of the percolation process to SLE_6 takes place *locally uniformly* with respect to the shape of the domain D and the positions of the starting and ending points a and b on its boundary ∂D.

We will use this version of convergence to SLE_6 to identify the Continuum Nonsimple Loop process with the scaling limit of *all* critical percolation clusters. A detailed proof of statement (S) can be found in [12]. Although the convergence statement in (S) is stronger than those in [36, 37], we note that it is restricted to Jordan domains, a restriction not present in [36, 37].

Before concluding this section, we give one more definition. Consider the exploration path $\gamma = \gamma_{D,x,y}^\delta$ and the set $\Gamma(\gamma) = \Gamma_Y(\gamma) \cup \Gamma_B(\gamma)$. The set $D^\delta \setminus \Gamma(\gamma)$ is the union of its connected components (in the lattice sense), which are simply connected. If the domain D is large and the e-vertices $x_a, y_a \in \partial D^\delta$ are not too close to each other, then with high probability the exploration process inside D^δ will make large excursions into D^δ, so that $D^\delta \setminus \Gamma(\gamma)$ will have more than one component. Given a point $z \in \mathbb{C}$ contained in $D^\delta \setminus \Gamma(\gamma)$, we will denote by $D_{a,b}^\delta(z)$ the domain corresponding to the unique element of $D^\delta \setminus \Gamma(\gamma)$ that contains z (notice that for a deterministic $z \in D$, $D_{a,b}^\delta(z)$ is well defined with high probability for δ small, i.e., when $z \in D^\delta$ and $z \notin \Gamma(\gamma)$).

4.2 Discrete Loop Construction

Next, we show how to construct, by twice using the exploration process described in Section 4.1, a loop Λ along the edges of $\mathcal{H}$ corresponding to the external boundary of a monochromatic cluster contained in a large, simply connected, Jordan set D with monochromatic blue (say) boundary conditions (see Figures 7 and 8).

Consider the exploration path $\gamma = \gamma_{D,x,y}$ and the sets $\Gamma_Y(\gamma)$ and $\Gamma_B(\gamma)$ (see Figure 7). The set $D \setminus \{\Gamma_Y(\gamma) \cup \Gamma_B(\gamma)\}$ is the union of its connected components (in the lattice sense), which are simply connected. If the domain D is large and the e-vertices $x, y \in \partial D$ are chosen not too close to each other, with large probability the exploration process inside D will make large excursions into D, so that $D \setminus \{\Gamma_Y(\gamma) \cup \Gamma_B(\gamma)\}$ will have many components.

There are four types of components which may be usefully thought of in terms of their external site boundaries: (1) those components whose site boundary contains both sites in $\Gamma_Y(\gamma)$ and $\Delta_{y,x}D$, (2) the analogous components with $\Delta_{y,x}D$ replaced by $\Delta_{x,y}D$ and $\Gamma_Y(\gamma)$ by $\Gamma_B(\gamma)$, (3) those components whose site boundary only contains sites in $\Gamma_Y(\delta)$, and finally (4) the analogous components with $\Gamma_Y(\gamma)$ replaced by $\Gamma_B(\gamma)$.

Notice that the components of type 1 are the only ones with $\pm$ boundary conditions, while all other components have monochromatic s-boundaries. For a given component D' of type 1, we can identify the two edges that separate the yellow and blue portions of its s-boundary. The vertices x' and y' of $\mathcal{H}$ where those two edges intersect $\partial D'$ are e-vertices and are chosen to be the starting and ending points of the exploration path $\gamma_{D',x',y'}$ inside D'.

If $x'', y'' \in \partial D$ are respectively the ending and starting points of the left excursion $\mathcal{E}$ of $\gamma_{D,x,y}$ that "created" D', by pasting together $\mathcal{E}$ and $\gamma_{D',x',y'}$ with the help of the edges of ∂D contained between x' and x'' and between y' and y'', we get a loop Λ which corresponds to the boundary of a yellow cluster adjacent to $\partial_{y,x}D$ (see Figure 8). Notice that the path $\gamma_{D',x',y'}$ in general splits D' into various other domains, all of which have
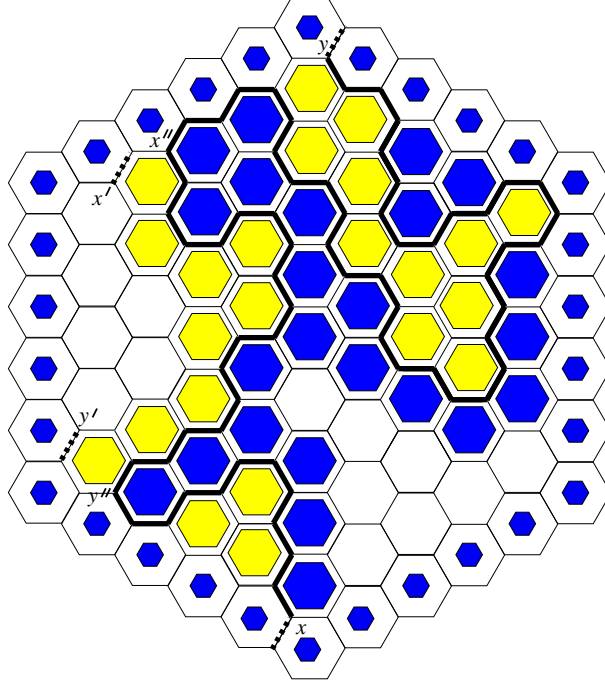
monochromatic boundary conditions.

Figure 7: First step of the construction of the outer contour of a cluster of yellow/minus (light in the figure) hexagons consisting of an exploration (heavy line) from the e-vertex x to the e-vertex y. The "starting edge" and "ending edge" of the exploration path are indicated by dotted segments next to x and y. The outer layer of hexagons does not belong to the domain where the explorations are carried out, but represents its monochromatic blue/plus external site boundary. x'' and y'' are the ending and starting points of a left excursion that determines a new domain D', and x' and y' are the vertices where the edges that separate the yellow and blue portions of the s-boundary of D' intersect $\partial D'$. x' and y' will be respectively the beginning and end of a new exploration path whose "starting edge" and "ending edge" are indicated by dotted segments next to those points.

4.3 Full Discrete Construction

We now give the algorithmic construction for discrete percolation which is the analogue of the continuum one. Each step of the construction is a single percolation exploration process; the order of successive steps is organized as in the continuum construction detailed in Section 3.2. We start with the largest Jordan set $D_0^\delta = \mathbb{D}^\delta$ of hexagons that is contained in the unit disc $\mathbb{D}$. We will also make use of the countable set $\mathcal{P}$ of points dense in $\mathbb{C}$ that was introduced earlier.

The first step consists of an exploration process inside D_0^δ. For this, we need to select two points x and y in ∂D_0^δ (which identify the starting and ending edges). We choose
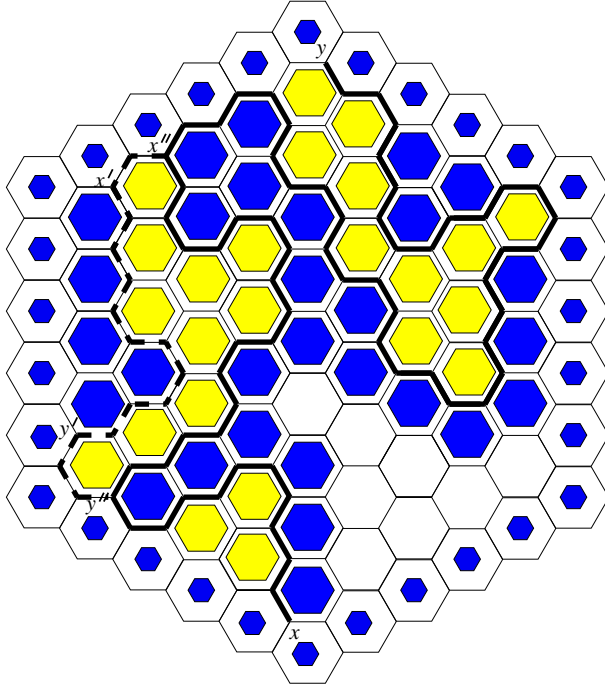
Figure 8: Second step of the construction of the outer contour of a cluster of yellow/minus (light in the figure) hexagons consisting of an exploration from x' to y' whose resulting path (heavy broken line) is pasted to the left excursion generated by the previous exploration with the help of edges (indicated again by a heavy broken line) of ∂D contained between x' and x'' and between y' and y''.

for x the e-vertex closest to $-i$, and for y the e-vertex closest to i (if there are two such vertices closest to $-i$, we can choose, say, the one with smallest real part, and analogously for i). The first exploration produces a path γ_1^δ and, for δ small, many new domains of all four types. These domains are ordered according to the maximal x- or y- distance d_m between points on their boundaries and, if necessary, with the help of points in $\mathcal{P}$, as in the continuum case, and that order is used, at each step of the construction, to determine the next exploration process. With this choice, the exploration processes and paths are naturally ordered: $\gamma_1^\delta, \gamma_2^\delta, \ldots$.

Each exploration process of course requires choosing a starting and ending vertex and edge. For domains of type 1, with a $\pm$ or $\mp$ boundary condition, the choice is the natural one, explained before.

For a domain D_k^δ (used at the kth step) of type other than 1, and therefore with a monochromatic boundary, the starting and ending edges are chosen with a procedure that mimics what is done in the continuum case. Once again, the exact procedure used to choose the pair of points is not important, as long as they are not chosen too close to each other. This is clear in the discrete case because the procedure that we are presenting is only "discovering" the cluster boundaries. In more precise terms, it is clear that one

could couple the processes obtained with different rules by means of the same percolation configuration, thus obtaining exactly the same cluster boundaries.

As in the continuum case, we can choose the following procedure. (In Theorem 5 we will slightly reorganize the procedure by using a coupling to the continuum construction to guarantee that the order of exploration of domains of the discrete and continuum procedures match despite the rules for breaking ties.) Given a domain D, x and y are chosen so that, of all pairs (u, v) of points in ∂D, they maximize $|\text{Re}(u-v)|$ if $|\text{Re}(u-v)| \geq |\text{Im}(u - v)|$, or else they maximize $|\text{Im}(u - v)|$. If the choice is not unique, to restrict the number of pairs one looks at those pairs, among the ones already obtained, that maximize the other of $\{|\text{Re}(u-v)|, |\text{Im}(u-v)|\}$. Notice that this leaves at most two pairs of points; if that's the case, the pair that contains the point with minimal real (and, if necessary, imaginary) part is chosen.

The procedure continues iteratively, with regions that have monochromatic boundaries playing the role played in the first step by the unit disc. Every time a region with $\pm$ boundary conditions is used, a new loop, corresponding to the outer boundary contour of a cluster, is formed by pasting together, as explained in Section 3.1, the new exploration path and the excursion containing the region where the last exploration was carried out. All the new regions created at a step when a loop is formed have monochromatic boundary conditions.

5   Main Technical Results

In this section we collect our main results about the Continuum Nonsimple Loop process. Before doing that, we state a precise version, called statement (S), of convergence of exploration paths to SLE_6 that we will use in the proofs of these results, presented in Section 6. Statement (S) is an immediate consequence of Theorem 5 of [12]. The proof given in [12], which relies among other things on the result of Smirnov [36] concerning convergence of crossing probabilities to Cardy's formula [13, 14], is an expanded and corrected version of Appendix A of [11]. We note that (S) is both more general and more special than the convergence statements in [36, 37] — more general in that the domain can vary with δ as $\delta \to 0$, but more special in the restriction to Jordan domains.

Given a Jordan domain D with two distinct points $a, b \in \partial D$ on its boundary, let $\mu_{D,a,b}$ denote the law of $\gamma_{D,a,b}$, the trace of chordal SLE_6, and let $\mu_{D,a,b}^{\delta}$ denote the law of the percolation exploration path $\gamma_{D,a,b}^{\delta}$. Let W be the space of continuous curves inside D from a to b. We define $\rho(\mu_{D,a,b}, \mu_{D,a,b}^{\delta}) \equiv \inf\{\varepsilon > 0 : \mu_{D,a,b}(U) \leq \mu_{D,a,b}^{\delta}(\bigcup_{x \in U} B_{\text{d}}(x, \varepsilon)) + \varepsilon$ for all Borel $U \subset W\}$ (where $B_{\text{d}}(x, \varepsilon)$ denotes the open ball of radius ε centered at x in the metric (2)) and denote by $\text{d}_{\text{P}}(\mu_{D,a,b}, \mu_{D,a,b}^{\delta}) \equiv \max\{\rho(\mu_{D,a,b}, \mu_{D,a,b}^{\delta}), \rho(\mu_{D,a,b}^{\delta}, \mu_{D,a,b})\}$ the Prohorov distance; weak convergence is equivalent to convergence in the Prohorov metric. Statement (S) is the following; it is used in the proofs of all the results of this section *except* for Lemmas 5.1-5.2.

(S) For Jordan domains, there is convergence in distribution of the percolation exploration path to the trace of chordal SLE_6 that is *locally uniform* in the shape

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of the boundary with respect to the uniform metric on continuous curves (2), and in the location of the starting and ending points with respect to the Euclidean metric; i.e., for (D, a, b) a Jordan domain with distinct $a, b \in \partial D$, $\forall \varepsilon > 0$, $\exists \alpha_0 = \alpha_0(\varepsilon)$ and $\delta_0 = \delta_0(\varepsilon)$ such that for all (D', a', b') with D' Jordan and with $\max\left(\mathrm{d}(\partial D, \partial D'), |a - a'|, |b - b'|\right) \leq \alpha_0$ and $\delta \leq \delta_0$, $\mathrm{d}_{\mathrm{P}}(\mu_{D', a', b'}, \mu_{D', a', b'}^{\delta}) \leq \varepsilon$.

5.1 Preliminary Results

We first give some important results which are needed in the proofs of the main theorems. We start with two lemmas which are consequences of [3], of standard bounds on the probability of events corresponding to having a certain number of monochromatic crossings of an annulus (see Lemma 5 of [21], Appendix A of [26], and also [5]), but which do *not* depend on statement (S).

Lemma 5.1. *Let $\gamma_{\mathbb{D}, -i, i}^{\delta}$ be the percolation exploration path on the edges of $\delta \mathcal{H}$ inside (the δ-approximation of) $\mathbb{D}$ between (the e-vertices closest to) $-i$ and i. For any fixed point $z \in \mathbb{D}$, chosen independently of $\gamma_{\mathbb{D}, -i, i}^{\delta}$, as $\delta \to 0$, $\gamma_{\mathbb{D}, -i, i}^{\delta}$ and the boundary $\partial \mathbb{D}_{-i, i}^{\delta}(z)$ of the domain $\mathbb{D}_{-i, i}^{\delta}(z)$ that contains z jointly have limits in distribution along subsequences of δ with respect to the uniform metric (2) on continuous curves. Moreover, any subsequence limit of $\partial \mathbb{D}_{-i, i}^{\delta}(z)$ is almost surely a simple loop [5].*

Lemma 5.2. *Using the notation of Lemma 5.1, let $\gamma_{\mathbb{D}, -i, i}$ be the limit in distribution of $\gamma_{\mathbb{D}, -i, i}^{\delta}$ as $\delta \to 0$ along some convergent subsequence $\{\delta_k\}$ and $\partial \mathbb{D}_{-i, i}(z)$ the boundary of the domain $\mathbb{D}_{-i, i}(z)$ of $\mathbb{D} \setminus \gamma_{D, -i, i}[0, 1]$ that contains z. Then, as $k \to \infty$, $(\gamma_{\mathbb{D}, -i, i}^{\delta_k}, \partial \mathbb{D}_{-i, i}^{\delta_k}(z))$ converges in distribution to $(\gamma_{\mathbb{D}, -i, i}, \partial \mathbb{D}_{-i, i}(z))$.*

The two lemmas above are important ingredients in the proof of Theorem 5 below. The second one says that, for every subsequence limit, the discrete boundaries converge to the boundaries of the domains generated by the limiting continuous curve. If we use statement (S), then the limit $\gamma_{\mathbb{D}, -i, i}$ of $\gamma_{\mathbb{D}, -i, i}^{\delta_k}$ is the trace of chordal SLE_6 for every subsequence $\delta_k \downarrow 0$, and we can use Lemmas 5.2 and 5.1 to deduce that all the domains produced in the continuum construction are Jordan domains. The key step in that direction is represented by the following result, our proof of which relies on (S).

Corollary 5.1. *For any deterministic $z \in \mathbb{D}$, the boundary $\partial \mathbb{D}_{-i, i}(z)$ of a domain $\mathbb{D}_{-i, i}(z)$ of the continuum construction is almost surely a Jordan curve.*

The corollary says that the domains that appear after the first step of the continuum construction are Jordan domains. The steps in the second stage of the continuum construction consist of SLE_6 paths inside Jordan domains, and therefore Corollary 5.1, combined with Riemann's mapping theorem and the conformal invariance of SLE_6, implies that the domains produced during the second stage are also Jordan. By induction, we deduce that all the domains produced in the continuum construction are Jordan domains.

We end this section with one more lemma which is another key ingredient in the proof of Theorem 5; we remark that its proof requires (S) in a fundamental way.

Lemma 5.3. *Let (D, a, b) denote a* random *Jordan domain, with a, b two points on ∂D. Let $\{(D_k, a_k, b_k)\}_{k \in \mathbb{N}}$, $a_k, b_k \in \partial D_k$, be a sequence of* random *Jordan domains with points on their boundaries such that, as $k \to \infty$, $(\partial D_k, a_k, b_k)$ converges in distribution to $(\partial D, a, b)$ with respect to the uniform metric (2) on continuous curves, and the Euclidean metric on (a, b). For any sequence $\{\delta_k\}_{k \in \mathbb{N}}$ with $\delta_k \downarrow 0$ as $k \to \infty$, $\gamma_{D_k, a_k, b_k}^{\delta_k}$ converges in distribution to $\gamma_{D, a, b}$ with respect to the uniform metric (2) on continuous curves.*

5.2 Main Technical Theorems

In this section we state the main technical theorems of this paper. Our main results, presented in Section 1.1, are consequences of these theorems. The proofs of these theorems rely on statement (S). As noted before, a detailed proof of statement (S) can be found in [12].

Theorem 5. *For any $k \in \mathbb{N}$, the first k steps of (a suitably reorganized version of) the full discrete construction inside the unit disc (of Section 4.3) converge, jointly in distribution, to the first k steps of the full continuum construction inside the unit disc (of Section 3.2). Furthermore, the scaling limit of the full (original or reorganized) discrete construction is the full continuum construction.*

Moreover, if for any fixed $\varepsilon > 0$ we let $K_\delta(\varepsilon)$ denote the number of steps needed to find all the cluster boundaries of Euclidean diameter larger than ε in the discrete construction, then $K_\delta(\varepsilon)$ is bounded in probability as $\delta \to 0$; i.e., $\lim_{C \to \infty} \limsup_{\delta \to 0} \mathbb{P}(K_\delta(\varepsilon) > C) = 0$. This is so in both the original and reorganized versions of the discrete construction.

The second part of Theorem 5 means that both versions of the discrete construction used in the theorem find all large contours in a number of steps which does not diverge as $\delta \to 0$. This, together with the first part of the same theorem, implies that the continuum construction does indeed describe all macroscopic contours contained inside the unit disc (with blue boundary conditions) as $\delta \to 0$.

The construction presented in Section 3.2 can of course be repeated for the disc $\mathbb{D}_R$ of radius R, for any R, so we should take a "thermodynamic limit" by letting $R \to \infty$. In this way, we would eliminate the boundary (and the boundary conditions) and obtain a process on the whole plane. Such an extension from the unit disc to the plane is contained in the next theorem.

Let P_R be the (limiting) distribution of the set of curves (all continuum nonsimple loops) generated by the continuum construction inside $\mathbb{D}_R$ (i.e., the limiting measure, defined by the inductive construction, on the complete separable metric space Ω_R of collections of continuous curves in $\mathbb{D}_R$).

For a domain D, we denote by I_D the mapping (on Ω or Ω_R) in which all portions of curves that exit D are removed. When applied to a configuration of loops in the plane, I_D gives a set of curves which either start and end at points on ∂D or form closed loops completely contained in D. Let $\hat{I}_D$ be the same mapping lifted to the space of probability measures on Ω or Ω_R.

Theorem 6. *There exists a unique probability measure P on the space Ω of collections of continuous curves in $\mathbb{R}^2$ such that $P_R \to P$ as $R \to \infty$ in the sense that for every bounded domain D, as $R \to \infty$, $\hat{I}_D P_R \to \hat{I}_D P$.*

Remark 5.1. *We we will generally take monochromatic blue boundary conditions on the disc $\mathbb{D}_R$ of radius R, but this arbitrary choice does not affect the results.*

The next theorem states a conformal invariance property of the Continuum Nonsimple Loop processes of Theorem 1.

Theorem 7. *Given two disjoint discs, D_1 and D_2, let λ_1 (respectively, λ_2) be the smallest loop from the Continuum Nonsimple Loop process X that surrounds D_1 (resp., D_2) and let $\tilde{D}_1$ (resp., $\tilde{D}_2$) be the connected component of $\mathbb{R}^2 \setminus \lambda_1$ (resp., $\mathbb{R}^2 \setminus \lambda_2$) that contains D_1 (resp., D_2). Assume that $\tilde{D}_1$ and $\tilde{D}_2$ are disjoint and let $P_{\tilde{D}_i}$, $i = 1, 2$, denote the distribution of the loops inside $\tilde{D}_i$. Then, conditioned on $\tilde{D}_1$ and $\tilde{D}_2$, the configurations inside $\tilde{D}_1$ and $\tilde{D}_2$ are independent and moreover $P_{\tilde{D}_2} = f * P_{\tilde{D}_1}$ (here $f * P_{\tilde{D}_1}$ denotes the probability distribution of the loop process $f(X')$ when X' is distributed by $P_{\tilde{D}_1}$), where $f : \tilde{D}_1 \to \tilde{D}_2$ is a conformal homeomorphism from $\tilde{D}_1$ onto $\tilde{D}_2$.*

We remark that the result is still valid (without the independence) even if $\tilde{D}_1$ and $\tilde{D}_2$ are not disjoint, but for simplicity we do not consider that case.

To conclude this section, we show how to recover chordal SLE_6 from the Continuum Nonsimple Loop process, i.e., given a (deterministic) Jordan domain D with two boundary points a and b, we give a construction that uses the continuum nonsimple loops of P to generate a process distributed like chordal SLE_6 inside D from a to b.

Remember, first of all, that each continuum nonsimple loop has either a clockwise or counterclockwise direction, with the set of all loops surrounding any deterministic point alternating in direction. For convenience, let us suppose that a is at the "bottom" and b is at the "top" of D so that the boundary is divided into a left and right part by these two points. Fix $\varepsilon > 0$ and call $LR(\varepsilon)$ the set of all the directed segments of loops that connect from the left to the right part of the boundary touching ∂D at a distance larger than ε from both a and b, and $RL(\varepsilon)$ the analogous set of directed segments from the right to the left portion of ∂D. For a fixed $\varepsilon > 0$, there is only a finite number of such segments, and, if they are ordered moving along the left boundary of D from a to b, they alternate in direction (i.e., a segment in $LR(\varepsilon)$ is followed by one in $RL(\varepsilon)$ and so on).

Between a segment in $RL(\varepsilon)$ and the next segment in $LR(\varepsilon)$, there are countably many portions of loops intersecting D which start and end on ∂D and are maximal in the sense that they are not contained inside any other portion of loop of the same type; they all have counterclockwise direction and can be used to make a "bridge" between the right-to-left segment and the next one (in $LR(\varepsilon)$). This is done by pasting the portions of loops together with the help of points in ∂D and a limit procedure to produce a connected (nonsimple) path.

If we do this for each pair of successive segments on both sides of the boundary of D, we get a path that connects two points on ∂D. By letting $\varepsilon \to 0$ and taking the

limit of this procedure, since almost surely a and b are surrounded by an infinite family of nested loops with diameters going to zero, we obtain a path that connects a with b; this path is distributed as chordal SLE_6 inside D from a to b. The last claim follows from considering the analogous procedure for percolation on the discrete lattice $\delta\mathcal{H}$, using segments of boundaries. It is easy to see that in the discrete case this procedure produces exactly the same path as the percolation exploration process. By Theorems 1 and 3, the scaling limit of this discrete procedure is the continuum one described above, therefore the claim follows from (S).

6 Proofs

In this section we present the proofs of the results stated in Sections 1.1 and 5. In order to do that, we will use the following lemma.

Lemma 6.1. Let $\mathcal{A}^\delta(v; \varepsilon, \varepsilon')$ be the event that the annulus $B(v, \varepsilon) \setminus B(v, \varepsilon')$ centered at $v \in \mathbb{D}$ contains six disjoint monochromatic crossings, not all of the same color, and let $\mathcal{B}^\delta(v; \varepsilon, \varepsilon')$ be the event, for some $v \in \partial\mathbb{D}$, that $\mathbb{D} \cap \{B(v, \varepsilon) \setminus B(v, \varepsilon')\}$ contains three disjoint monochromatic crossings, not all of the same color. Then, for any $\varepsilon > 0$,

$$\lim_{\varepsilon' \to 0} \limsup_{\delta \to 0} \mathbb{P}(\bigcup_{v \in \mathbb{D}} \mathcal{A}^\delta(v; \varepsilon, \varepsilon')) = 0 \tag{9}$$

and

$$\lim_{\varepsilon' \to 0} \limsup_{\delta \to 0} \mathbb{P}(\bigcup_{v \in \partial\mathbb{D}} \mathcal{B}^\delta(v; \varepsilon, \varepsilon')) = 0. \tag{10}$$

Proof. We know from [21] that there exist $c_1 < \infty$ and $\alpha > 0$ so that for $\varepsilon_2 < \varepsilon_1$, and δ small enough (in particular, $\delta < \varepsilon_2$),

$$\mathbb{P}(\mathcal{A}^\delta(v; \varepsilon_1, \varepsilon_2)) \leq c_1 \left(\frac{\varepsilon_2}{\varepsilon_1}\right)^{2+\alpha} \tag{11}$$

for any $v \in \mathbb{R}^2$. If we cover $\mathbb{D}$ with $N_{\varepsilon'}$ balls of radius ε' centered at $\{v_j\}_{j \in \mathcal{N}_{\varepsilon'}}$, we have that, for $\varepsilon' < \varepsilon/6$ and δ small enough,

$$\mathbb{P}(\bigcup_{v \in \mathbb{D}} \mathcal{A}^\delta(v; \varepsilon, \varepsilon')) \leq \mathbb{P}(\bigcup_{j \in \mathcal{N}_{\varepsilon'}} \mathcal{A}^\delta(v_j; \varepsilon/2, 3\,\varepsilon')) \leq 6^{2+\alpha} \, c_1 \, N_{\varepsilon'} \left(\frac{\varepsilon'}{\varepsilon}\right)^{2+\alpha}, \tag{12}$$

where the first inequality follows from the observations that for any $v \in \mathbb{D}$, $B(v, \varepsilon') \subset B(v_j, 3\,\varepsilon')$ and $B(v_j, \varepsilon/2) \subset B(v_j, \varepsilon - \varepsilon') \subset B(v, \varepsilon)$ for some $j \in \mathcal{N}_{\varepsilon'}$, and the second inequality uses (11). Using the fact that $N_{\varepsilon'}$ is $O(\frac{1}{\varepsilon'})^2$, we can let first $\delta \to 0$ and then $\varepsilon' \to 0$ to obtain (9).

We also know, as a consequence of Lemma 5 of [21] or as proved in Appendix A of [26], that for any $v \in \mathbb{R}$, the probability that the semi-annulus $\mathbb{H} \cap \{B(v, \varepsilon_1) \setminus B(v, \varepsilon_2)\}$ contains

three disjoint monochromatic crossings, not all of the same color, is bounded above by $c_2 \, (\varepsilon_2/\varepsilon_1)^{1+\beta}$ for some $c_2 < \infty$ and $\beta > 0$. (We remark that the result still applies when $\mathbb{H}$ is replaced by any other half-plane.) Since the unit disc is a convex subset of the half-plane $\{x + iy : y > -1\}$ and therefore the intersection of an annulus centered at $-i$ with the unit disc $\mathbb{D}$ is a subset of the intersection of the same annulus with the half-plane $\{x + iy : y > -1\}$, we can use that bound to conclude that for $v = -i$, and in fact for any $v \in \partial\mathbb{D}$, there exists a constant $c_2 < \infty$ such that

$$\mathbb{P}(\mathcal{B}^\delta(v; \varepsilon_1, \varepsilon_2)) \leq c_2 \left(\frac{\varepsilon_2}{\varepsilon_1}\right)^{1+\beta} \tag{13}$$

for some $\beta > 0$. We can then use similar arguments to those above, together with (13), to obtain (10) and conclude the proof. $\square$

Proof of Lemma 5.1. The first part of the lemma is a direct consequence of [3]; it is enough to notice that the (random) polygonal curves $\gamma^\delta_{\mathbb{D},-i,i}$ and $\partial\mathbb{D}^\delta_{-i,i}(z)$ satisfy the conditions in [3] and thus have a scaling limit in terms of continuous curves, at least along subsequences of δ.

To prove the second part, we use a standard percolation bound (see Lemma 5 of [21]) to show that, in the limit $\delta \to 0$, the loop $\partial\mathbb{D}^\delta_{-i,i}(z)$ does not collapse on itself but remains a simple loop.

Let us assume that this is not the case and that the limit $\tilde{\gamma}$ of $\partial\mathbb{D}^{\delta_k}_{-i,i}(z)$ along some subsequence $\{\delta_k\}_{k \in \mathbb{N}}$ touches itself, i.e., $\tilde{\gamma}(t_0) = \tilde{\gamma}(t_1)$ for $t_0 \neq t_1$ with positive probability. If that happens, we can take $\varepsilon > \varepsilon' > 0$ small enough so that the annulus $B(\tilde{\gamma}(t_1), \varepsilon) \setminus B(\tilde{\gamma}(t_1), \varepsilon')$ is crossed at least four times by $\tilde{\gamma}$ (here $B(u, r)$ is the ball of radius r centered at u).

Because of the choice of topology, the convergence in distribution of $\partial\mathbb{D}^{\delta_k}_{-i,i}(z)$ to $\tilde{\gamma}$ implies that we can find coupled versions of $\partial\mathbb{D}^{\delta_k}_{-i,i}(z)$ and $\tilde{\gamma}$ on some probability space $(\Omega', \mathcal{B}', \mathbb{P}')$ such that $\mathrm{d}(\partial\mathbb{D}^\delta_{-i,i}(z), \tilde{\gamma}) \to 0$, for all $\omega' \in \Omega'$ as $k \to \infty$ (see, for example, Corollary 1 of [9]).

Using this coupling, we can choose k large enough (depending on ω') so that $\partial\mathbb{D}^{\delta_k}_{-i,i}(z)$ stays in an $\varepsilon'/2$-neighborhood $\mathcal{N}(\tilde{\gamma}, \varepsilon'/2) \equiv \bigcup_{u \in \tilde{\gamma}} B(u, \varepsilon'/2)$ of $\tilde{\gamma}$. This event however would correspond to (at least) four paths of one color (corresponding to the four crossings by $\Delta\mathbb{D}^{\delta_k}_{-i,i}(z)$, which shadows $\partial\mathbb{D}^{\delta_k}_{-i,i}(z)$) and two of the other color (belonging to percolation clusters adjacent to the cluster of $\Delta\mathbb{D}^{\delta_k}_{-i,i}(z)$, and of the opposite color), of the annulus $B(\tilde{\gamma}(t_1), \varepsilon - \varepsilon'/2) \setminus B(\tilde{\gamma}(t_1), 3\,\varepsilon'/2)$ (see, for example, [5] — see also Figure 9). As $\delta_k \to 0$, we can let $\varepsilon' \to 0$, in which case the probability of seeing the event just described somewhere inside $\mathbb{D}$ goes to zero by an application of Lemma 6.1, leading to a contradiction. $\square$

In order to prove Lemma 5.2, we will use the following result.

Lemma 6.2. *For two (deterministic) points $u, v \in \mathbb{D}$, the probability that $\mathbb{D}_{-i,i}(u) = \mathbb{D}_{-i,i}(v)$ but $\mathbb{D}^\delta_{-i,i}(u) \neq \mathbb{D}^\delta_{-i,i}(v)$ or vice versa goes to zero as $\delta \to 0$.*
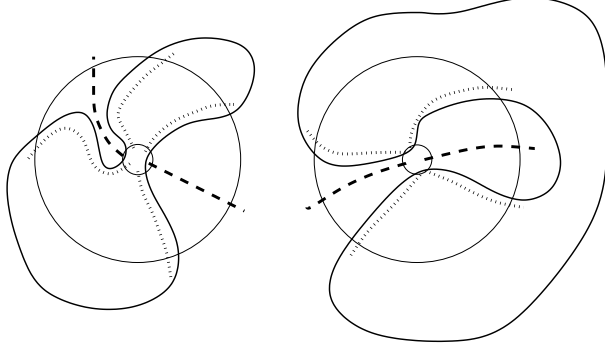
Figure 9: Schematic diagrams representing four blue (dotted in the figure) and two yellow (dashed in the figure) crossings of an annulus produced by having four crossings of the same annulus by a boundary (the solid loops).

Proof. Let $\{\delta_k\}_{k\in\mathbb{N}}$ be a convergent subsequence for $\gamma^\delta_{\mathbb{D},-i,i}$ and let $\gamma \equiv \gamma_{\mathbb{D},-i,i}$ be the limit in distribution of $\gamma^{\delta_k}_{\mathbb{D},-i,i}$ as $k \to \infty$. For simplicity of notation, in the rest of the proof we will drop the k and write δ instead of δ_k. Because of the choice of topology, the convergence in distribution of $\gamma^\delta \equiv \gamma^\delta_{\mathbb{D},-i,i}$ to γ implies that we can find coupled versions of γ^δ and γ on some probability space $(\Omega', \mathcal{B}', \mathbb{P}')$ such that $\mathrm{d}(\gamma^\delta(\omega'), \gamma(\omega')) \to 0$, for all ω' as $k \to \infty$ (see, for example, Corollary 1 of [9]).

Using this coupling, we first consider the case of u, v such that $\mathbb{D}_{-i,i}(u) = \mathbb{D}_{-i,i}(v)$ but $\mathbb{D}^\delta_{-i,i}(u) \neq \mathbb{D}^\delta_{-i,i}(v)$. Since $\mathbb{D}_{-i,i}(u)$ is an open subset of $\mathbb{C}$, there exists a continuous curve $\gamma_{u,v}$ joining u and v and a constant $\varepsilon > 0$ such that the ε-neighborhood $\mathcal{N}(\gamma_{u,v}, \varepsilon)$ of the curve is contained in $\mathbb{D}_{-i,i}(u)$, which implies that γ does not intersect $\mathcal{N}(\gamma_{u,v}, \varepsilon)$. Now, if γ^δ does not intersect $\mathcal{N}(\gamma_{u,v}, \varepsilon/2)$, for δ small enough, then there is a $\mathcal{T}$-path π of unexplored hexagons connecting the hexagon that contains u with the hexagon that contains v, and we conclude that $\mathbb{D}^\delta_{-i,i}(u) = \mathbb{D}^\delta_{-i,i}(v)$.

This shows that the event that $\mathbb{D}_{-i,i}(u) = \mathbb{D}_{-i,i}(v)$ but $\mathbb{D}^\delta_{-i,i}(u) \neq \mathbb{D}^\delta_{-i,i}(v)$ implies the existence of a curve $\gamma_{u,v}$ whose ε-neighborhood $\mathcal{N}(\gamma_{u,v}, \varepsilon)$ is not intersected by γ but whose $\varepsilon/2$-neighborhood $\mathcal{N}(\gamma_{u,v}, \varepsilon/2)$ is intersected by γ^δ. This implies that $\forall u, v \in \mathbb{D}$, $\exists \varepsilon > 0$ such that $\mathbb{P}'(\mathbb{D}_{-i,i}(u) = \mathbb{D}_{-i,i}(v)$ but $\mathbb{D}^\delta_{-i,i}(u) \neq \mathbb{D}^\delta_{-i,i}(v)) \leq \mathbb{P}'(\mathrm{d}(\gamma^\delta, \gamma) \geq \varepsilon/2)$. But the right hand side goes to zero for every $\varepsilon > 0$ as $\delta \to 0$, which concludes the proof of one direction of the claim.

To prove the other direction, we consider two points $u, v \in D$ such that $D_{-i,i}(u) \neq D_{-i,i}(v)$ but $D^\delta_{-i,i}(u) = D^\delta_{-i,i}(v)$. Assume that u is trapped before v by γ and suppose for the moment that $\mathbb{D}_{-i,i}(u)$ is a domain of type 3 or 4; the case of a domain of type 1 or 2 is analogous and will be treated later. Let t_1 be the first time u is trapped by γ with $\gamma(t_0) = \gamma(t_1)$ the double point of γ where the domain $\mathbb{D}_{-i,i}(u)$ containing u is "sealed off." At time t_1, a new domain containing u is created and v is disconnected from u.

Choose $\varepsilon > 0$ small enough so that neither u nor v is contained in the ball $B(\gamma(t_1), \varepsilon)$ of radius ε centered at $\gamma(t_1)$, nor in the ε-neighborhood $\mathcal{N}(\gamma[t_0, t_1], \varepsilon)$ of the portion of

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γ which surrounds u. Then it follows from the coupling that, for δ small enough, there are appropriate parameterizations of γ and γ^δ such that the portion $\gamma^\delta[t_0, t_1]$ of $\gamma^\delta(t)$ is inside $\mathcal{N}(\gamma[t_0, t_1], \varepsilon)$, and $\gamma^\delta(t_0)$ and $\gamma^\delta(t_1)$ are contained in $B(\gamma(t_1), \varepsilon)$.

For u and v to be contained in the same domain in the discrete construction, there must be a $\mathcal{T}$-path π of unexplored hexagons connecting the hexagon that contains u to the hexagon that contains v. ¿From what we said in the previous paragraph, any such $\mathcal{T}$-path connecting u and v would have to go though a "bottleneck" in $B(\gamma(t_1), \varepsilon)$ (see Figure 10).

Assume now, for concreteness but without loss of generality, that $\mathbb{D}_{-i,i}(u)$ is a domain of type 3, which means that γ winds around u counterclockwise, and consider the hexagons to the "left" of $\gamma^\delta[t_0, t_1]$ (these are all lightly shaded in Figure 10). Those hexagons form a "quasi-loop" around u since they wind around it (counterclockwise) and the first and last hexagons are both contained in $B(\gamma(t_1), \varepsilon)$. The hexagons to the left of $\gamma^\delta[t_0, t_1]$ belong to the set $\Gamma_Y(\gamma^\delta)$, which can be seen as a (nonsimple) path by connecting the centers of the hexagons in $\Gamma_Y(\gamma^\delta)$ by straight segments. Such a path shadows γ^δ, with the difference that it can have double (or even triple) points, since the same hexagon can be visited more than once. Consider $\Gamma_Y(\gamma^\delta)$ as a path $\hat{\gamma}^\delta$ with a given parametrization $\hat{\gamma}^\delta(t)$, chosen so that $\hat{\gamma}^\delta(t)$ is inside $B(\gamma(t_1), \varepsilon)$ when $\gamma^\delta(t)$ is, and it winds around u together with $\gamma^\delta(t)$.
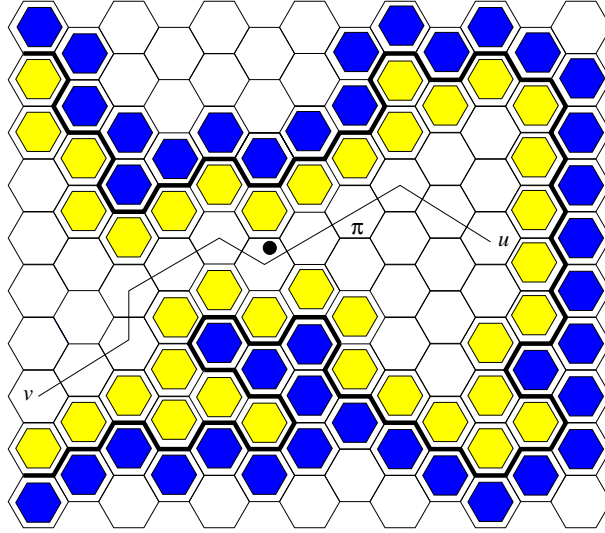

Figure 10: Example of a $\mathcal{T}$-path π of unexplored hexagons from u to v having to go through a "bottleneck" due to the fact that the exploration path (heavy line) comes close to itself. An approximate location of the continuum double point at $\gamma(t_0) = \gamma(t_1)$ is indicated by the small disc in one of the hexagons in the bottleneck area.

Now suppose that there were two times, $\hat{t}_0$ and $\hat{t}_1$, such that $\hat{\gamma}^\delta(\hat{t}_1) = \hat{\gamma}^\delta(\hat{t}_0) \in B(\gamma(t_1), \varepsilon)$ and $\hat{\gamma}^\delta[\hat{t}_0, \hat{t}_1]$ winds around u. This would imply that the "quasi-loop" of explored yellow hexagons around u is actually completed, and that $D_{a,b}^\delta(v) \neq D_{a,b}^\delta(u)$. Thus, for u and v to belong to the same discrete domain, this cannot happen.

For any $0 < \varepsilon' < \varepsilon$, if we take δ small enough, $\hat{\gamma}^\delta$ will be contained inside $\mathcal{N}(\gamma, \varepsilon')$, due to the coupling. Following the considerations above, the fact that u and v belong to the same domain in the discrete construction but to different domains in the continuum construction implies, for δ small enough, that there are four disjoint yellow $\mathcal{T}$-paths crossing the annulus $B(\gamma(t_1), \varepsilon) \setminus B(\gamma(t_1), \varepsilon')$ (the paths have to be disjoint because, as we said, $\hat{\gamma}^\delta$ cannot, when coming back to $B(\gamma(t_1), \varepsilon)$ after winding around u, touch itself inside $B(\gamma(t_1), \varepsilon)$). Since $B(\gamma(t_1), \varepsilon) \setminus B(\gamma(t_1), \varepsilon')$ is also crossed by at least two blue $\mathcal{T}$-paths from $\Gamma_B(\gamma^\delta)$, there is a total of at least six $\mathcal{T}$-paths, not all of the same color, crossing the annulus $B(\gamma(t_1), \varepsilon) \setminus B(\gamma(t_1), \varepsilon')$. We can then use Lemma 6.1 to conclude that, if we keep ε fixed and let $\delta \to 0$ and $\varepsilon' \to 0$, the probability to see such an event anywhere in $\mathbb{D}$ goes to zero.

In the case in which u belongs to a domain of type 1 or 2, let $\mathcal{E}$ be the excursion that traps u and $\gamma(t_0) \in \partial\mathbb{D}$ be the point on the boundary of $\mathbb{D}$ where $\mathcal{E}$ starts and $\gamma(t_1) \in \partial\mathbb{D}$ the point where it ends. Choose $\varepsilon > 0$ small enough so that neither u nor v is contained in the balls $B(\gamma(t_0), \varepsilon)$ and $B(\gamma(t_1), \varepsilon)$ of radius ε centered at $\gamma(t_0)$ and $\gamma(t_1)$, nor in the ε-neighborhood $\mathcal{N}(\mathcal{E}, \varepsilon)$ of the excursion $\mathcal{E}$. Because of the coupling, for δ small enough (depending on ε), γ^δ shadows γ along $\mathcal{E}$, staying within $\mathcal{N}(\mathcal{E}, \varepsilon)$. If this is the case, any $\mathcal{T}$-path of unexplored hexagons connecting the hexagon that contains u with the hexagon that contains v would have to go through one of two "bottlenecks," one contained in $B(\gamma(t_0), \varepsilon)$ and the other in $B(\gamma(t_1), \varepsilon)$.

Assume for concreteness (but without loss of generality) that u is in a domain of type 1, which means that γ winds around u counterclockwise. If we parameterize γ and γ^δ so that $\gamma^\delta(t_0) \in B(\gamma(t_0), \varepsilon)$ and $\gamma^\delta(t_1) \in B(\gamma(t_1), \varepsilon)$, $\gamma^\delta[t_0, t_1]$ forms a "quasi-excursion" around u since it winds around it (counterclockwise) and it starts inside $B_\varepsilon(\gamma(t_0))$ and ends inside $B_\varepsilon(\gamma(t_1))$. Notice that if γ^δ touched $\partial\mathbb{D}^\delta$, inside both $B_\varepsilon(\gamma(t_0))$ and $B_\varepsilon(\gamma(t_1))$, this would imply that the "quasi-excursion" is a real excursion and that $D_{a,b}^\delta(v) \neq D_{a,b}^\delta(u)$.

For any $0 < \varepsilon' < \varepsilon$, if we take δ small enough, γ^δ will be contained inside $\mathcal{N}(\gamma, \varepsilon')$, due to the coupling. Therefore, the fact that $\mathbb{D}_{a,b}^\delta(v) = \mathbb{D}_{a,b}^\delta(u)$ implies, with probability going to one as $\delta \to 0$, that for $\varepsilon > 0$ fixed and any $0 < \varepsilon' < \varepsilon$, γ^δ enters the ball $B(\gamma(t_i), \varepsilon')$ and does not touch $\partial\mathbb{D}^\delta$ inside the larger ball $B(\gamma(t_i), \varepsilon)$, for $i = 0$ or 1. This is equivalent to having at least two yellow and one blue $\mathcal{T}$-paths (contained in $\mathbb{D}^\delta$) crossing the annulus $B(\gamma(t_i), \varepsilon) \setminus B(\gamma(t_i), \varepsilon')$. As $\delta \to 0$, we can let ε' go to zero (keeping ε fixed) and use Lemma 6.1 to conclude that the probability that such an event occurs anywhere on the boundary of the unit disc goes to zero.

We have shown that, for two fixed points $u, v \in \mathbb{D}$, having $\mathbb{D}_{-i,i}(u) \neq \mathbb{D}_{-i,i}(v)$ but $\mathbb{D}_{-i,i}^\delta(u) = \mathbb{D}_{-i,i}^\delta(v)$ or vice versa implies the occurrence of an event whose probability goes to zero as $\delta \to 0$, and the proof of the lemma is concluded. $\square$

Proof of Lemma 5.2. As in the proof of Lemma 6.2, we let $\{\delta_k\}_{k \in \mathbb{N}}$ be a convergent subsequence for $\gamma_{\mathbb{D}, -i, i}^\delta$ and let $\gamma \equiv \gamma_{\mathbb{D}, -i, i}$ be the limit in distribution of $\gamma_{\mathbb{D}, -i, i}^{\delta_k}$ as $k \to \infty$, and in the rest of the proof consider coupled versions of $\gamma^{\delta_k} \equiv \gamma_{\mathbb{D}, -i, i}^{\delta_k}$ and γ.

Let us introduce the Hausdorff distance $d_H(A, B)$ between two closed nonempty subsets

of $\overline{\mathbb{D}}$:

$$\mathrm{d}_{\mathrm{H}}(A, B) \equiv \inf\{\ell \geq 0 : B \subset \cup_{a \in A} B(a, \ell),\ A \subset \cup_{b \in B} B(b, \ell)\}. \tag{14}$$

With this metric, the collection of closed subsets of $\overline{\mathbb{D}}$ is a compact space. We will next prove that $\partial \mathbb{D}_{-i,i}^{\delta_k}(z)$ converges in distribution to $\partial \mathbb{D}_{-i,i}(z)$ as $\delta_k \to 0$, in the topology induced by (14). (Notice that the coupling between γ^{δ_k} and γ provides a coupling between $\partial \mathbb{D}_{-i,i}^{\delta_k}(z)$ and $\partial \mathbb{D}_{-i,i}(z)$, seen as boundaries of domains produced by the two paths.)

We will now use Lemma 5.1 and take a further subsequence k_n of the δ_k's that for simplicity of notation we denote by $\{\delta_n\}_{n \in \mathbb{N}}$ such that, as $n \to \infty$, $\{\gamma^{\delta_n}, \partial \mathbb{D}_{-i,i}^{\delta_n}(z)\}$ converge jointly in distribution to $\{\gamma, \tilde{\gamma}\}$, where $\tilde{\gamma}$ is a simple loop. For any $\varepsilon > 0$, since $\tilde{\gamma}$ is a compact set, we can find a covering of $\tilde{\gamma}$ by a finite number of balls of radius $\varepsilon/2$ centered at points on $\tilde{\gamma}$. Each ball contains both points in the interior $\mathrm{int}(\tilde{\gamma})$ of $\tilde{\gamma}$ and in the exterior $\mathrm{ext}(\tilde{\gamma})$ of $\tilde{\gamma}$, and we can choose (independently of n) one point from $\mathrm{int}(\tilde{\gamma})$ and one from $\mathrm{ext}(\tilde{\gamma})$ inside each ball.

Once again, the convergence in distribution of $\partial \mathbb{D}_{-i,i}^{\delta_n}(z)$ to $\tilde{\gamma}$ implies the existence of a coupling such that, for n large enough, the selected points that are in $\mathrm{int}(\tilde{\gamma})$ are contained in $\mathbb{D}_{-i,i}^{\delta_n}(z)$, and those that are in $\mathrm{ext}(\tilde{\gamma})$ are contained in the complement of $\overline{\mathbb{D}_{-i,i}^{\delta_n}(z)}$. But by Lemma 6.2, each one of the selected points that is contained in $\mathbb{D}_{-i,i}^{\delta_n}(z)$ is also contained in $\mathbb{D}_{-i,i}(z)$ with probability going to 1 as $n \to \infty$; analogously, each one of the selected points contained in the complement of $\overline{\mathbb{D}_{-i,i}^{\delta_n}(z)}$ is also contained in the complement of $\overline{\mathbb{D}_{-i,i}(z)}$ with probability going to 1 as $n \to \infty$. This implies that $\partial \mathbb{D}_{-i,i}(z)$ crosses each one of the balls in the covering of $\tilde{\gamma}$, and therefore $\tilde{\gamma} \subset \cup_{u \in \partial \mathbb{D}_{-i,i}(z)} B(u, \varepsilon)$. ¿From this and the coupling between $\partial \mathbb{D}_{-i,i}^{\delta_n}(z)$ and $\tilde{\gamma}$, it follows immediately that, for n large enough, $\partial \mathbb{D}_{-i,i}^{\delta_n}(z) \subset \cup_{u \in \partial \mathbb{D}_{-i,i}(z)} B(u, \varepsilon)$ with probability close to one.

A similar argument (analogous to the previous one but simpler, since it does not require the use of $\tilde{\gamma}$), with the roles of $\mathbb{D}_{-i,i}^{\delta_n}(z)$ and $\mathbb{D}_{-i,i}(z)$ inverted, shows that $\partial \mathbb{D}_{-i,i}(z) \subset \cup_{u \in \partial \mathbb{D}_{-i,i}^{\delta_n}(z)} B(u, \varepsilon)$ with probability going to 1 as $n \to \infty$. Therefore, for all $\varepsilon > 0$, $\mathbb{P}(\mathrm{d}_{\mathrm{H}}(\partial \mathbb{D}_{-i,i}^{\delta_n}(z), \partial \mathbb{D}_{-i,i}(z)) > \varepsilon) \to 0$ as $n \to \infty$, which implies convergence in distribution of $\partial \mathbb{D}_{-i,i}^{\delta_n}(z)$ to $\partial \mathbb{D}_{-i,i}(z)$, as $\delta_n \to 0$, in the topology induced by (14). But Lemma 5.1 implies that $\partial \mathbb{D}_{-i,i}^{\delta_n}(z)$ converges in distribution (using (2)) to a simple loop, therefore $\partial \mathbb{D}_{-i,i}(z)$ must also be a simple loop; and we have convergence in the topology induced by (2).

It is also clear that the argument above is independent of the subsequence $\{\delta_n\}$ (and of the original subsequence $\{\delta_k\}$), so the limit of $\partial \mathbb{D}_{-i,i}^{\delta}(z)$ is unique and coincides with $\partial \mathbb{D}_{-i,i}(z)$. Hence, we have convergence in distribution of $\partial \mathbb{D}_{-i,i}^{\delta}(z)$ to $\partial \mathbb{D}_{-i,i}(z)$, as $\delta \to 0$, in the topology induced by (2), and indeed joint convergence of $(\gamma^{\delta}, \partial \mathbb{D}_{-i,i}^{\delta}(z))$ to $(\gamma, \partial \mathbb{D}_{-i,i}(z))$. $\square$

Proof of Corollary 5.1. The corollary follows immediately from Lemma 5.1 and Lemma 5.2, as already seen in the proof of Lemma 5.2. $\square$

Proof of Lemma 5.3. First of all recall that the convergence of $(\partial D_k, a_k, b_k)$ to $(\partial D, a, b)$

in distribution implies the existence of coupled versions of $(\partial D_k, a_k, b_k)$ and $(\partial D, a, b)$ on some probability space $(\Omega', \mathcal{B}', \mathbb{P}')$ such that $\mathrm{d}(\partial D(\omega'), \partial D_k(\omega')) \to 0$, $a_k(\omega') \to a(\omega')$, $b_k(\omega') \to b(\omega')$ for all ω' as $k \to \infty$ (see, for example, Corollary 1 of [9]). This immediately implies that the conditions to apply Radó's theorem (see Theorem 4 of Section 2.5) are satisfied. Let f_k be the conformal map that takes the unit disc $\mathbb{D}$ onto D_k with $f_k(0) = 0$ and $f_k'(0) > 0$, and let f be the conformal map from $\mathbb{D}$ onto D with $f(0) = 0$ and $f'(0) > 0$. Then, by Theorem 4, f_k converges to f uniformly in $\overline{\mathbb{D}}$, as $k \to \infty$.

Let γ (resp., γ_k) be the chordal SLE_6 inside D (resp., D_k) from a to b (resp., from a_k to b_k), $\tilde{\gamma} = f^{-1}(\gamma)$, $\tilde{a} = f^{-1}(a)$, $\tilde{b} = f^{-1}(b)$, and $\tilde{\gamma}_k = f_k^{-1}(\gamma_k)$, $\tilde{a}_k = f_k^{-1}(a_k)$, $\tilde{b}_k = f_k^{-1}(b_k)$. We note that, because of the conformal invariance of chordal SLE_6, $\tilde{\gamma}$ (resp., $\tilde{\gamma}_k$) is distributed as chordal SLE_6 in $\mathbb{D}$ from $\tilde{a}$ to $\tilde{b}$ (resp., from $\tilde{a}_k$ to $\tilde{b}_k$). Since $|a - a_k| \to 0$ and $|b - b_k| \to 0$ for all ω', and $f_k \to f$ uniformly in $\overline{\mathbb{D}}$, we conclude that $|\tilde{a} - \tilde{a}_k| \to 0$ and $|\tilde{b} - \tilde{b}_k| \to 0$ for all ω'.

Later we will prove a "continuity" property of SLE_6 (Lemma 6.3) that allows us to conclude that, under these conditions, $\tilde{\gamma}_k$ converges in distribution to $\tilde{\gamma}$ in the uniform metric (2) on continuous curves. Once again, this implies the existence of coupled versions of $\tilde{\gamma}_k$ and $\tilde{\gamma}$ on some probability space $(\Omega', \mathcal{B}', \mathbb{P}')$ such that $\mathrm{d}(\tilde{\gamma}(\omega'), \tilde{\gamma}_k(\omega')) \to 0$, for all ω' as $k \to \infty$. Therefore, thanks to the convergence of f_k to f uniformly in $\overline{\mathbb{D}}$, $\mathrm{d}(f(\tilde{\gamma}(\omega')), f_k(\tilde{\gamma}_k(\omega'))) \to 0$, for all ω' as $k \to \infty$. But since $f_k(\tilde{\gamma}_k)$ is distributed as γ_{D_k, a_k, b_k} and $f(\tilde{\gamma})$ is distributed as $\gamma_{D, a, b}$, we conclude that, as $k \to \infty$, γ_{D_k, a_k, b_k} converges in distribution to $\gamma_{D, a, b}$ in the uniform metric (2) on continuous curves.

We now note that (S) implies that, as $\delta \to 0$, $\gamma_{D_k, a_k, b_k}^{\delta}$ converges in distribution to γ_{D_k, a_k, b_k} *uniformly* in k, for k large enough. Therefore, as $k \to \infty$, $\gamma_{D_k, a_k, b_k}^{\delta_k}$ converges in distribution to $\gamma_{D, a, b}$, and the proof is concluded. $\square$

Lemma 6.3. *Let $\mathbb{D} \subset \mathbb{C}$ be the unit disc, a and b two distinct points on its boundary, and γ the trace of chordal SLE_6 inside $\mathbb{D}$ from a to b. Let $\{a_k\}$ and $\{b_k\}$ be two sequences of points in $\partial \mathbb{D}$ such that $a_k \to a$ and $b_k \to b$. Then, as $k \to \infty$, the trace γ_k of chordal SLE_6 inside $\mathbb{D}$ from a_k to b_k converges in distribution to γ in the uniform topology (2) on continuous curves.*

Proof. Let $f_k(z) = e^{i\alpha_k} \frac{z - z_k}{1 - \bar{z}_k z}$ be the (unique) linear fractional transformation that takes the unit disc $\mathbb{D}$ onto itself, mapping a to a_k, b to b_k, and a third point $c \in \partial \mathbb{D}$ distinct from a and b to itself. α_k and z_k depend continuously on a_k and b_k. As $k \to \infty$, since $a_k \to a$ and $b_k \to b$, f_k converges uniformly to the identity in $\overline{\mathbb{D}}$.

Using the conformal invariance of chordal SLE_6, we couple γ_k and γ by writing $\gamma_k = f_k(\gamma)$. The uniform convergence of f_k to the identity implies that $\mathrm{d}(\gamma, \gamma_k) \to 0$ as $k \to \infty$, which is enough to conclude that γ_k converges to γ in distribution. $\square$

Proof of Theorem 5. Let us prove the second part of the theorem first. We will do this for the original version of the discrete construction, but essentially the same proof works for the reorganized version we will describe below, as we will explain later. Suppose that at step k of this discrete construction an exploration process γ_k^{δ} is run inside a domain D_{k-1}^{δ}, and write $D_{k-1}^{\delta} \setminus \Gamma(\gamma_k^{\delta}) = \bigcup_j D_{k,j}^{\delta}$, where $\{D_{k,j}^{\delta}\}$ are the maximal connected domains

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of unexplored hexagons into which D_{k-1}^δ is split by removing the set $\Gamma(\gamma_k^\delta)$ of hexagons explored by γ_k^δ.

Let $d_x(D_{k-1}^\delta)$ and $d_y(D_{k-1}^\delta)$ be respectively the maximal x- and y-distances between pairs of points in ∂D_{k-1}^δ. Suppose, without loss of generality, that $d_x(D_{k-1}^\delta) \geq d_y(D_{k-1}^\delta)$, and consider the rectangle $\mathcal{R}$ (see Figure 11) whose vertical sides are aligned to the y-axis, have length $d_x(D_{k-1}^\delta)$, and are each placed at x-distance $\frac{1}{3} d_x(D_{k-1}^\delta)$ from points of ∂D_{k-1}^δ with minimal or maximal x-coordinate in such a way that the horizontal sides of $\mathcal{R}$ have length $\frac{1}{3} d_x(D_{k-1}^\delta)$; the bottom and top sides of $\mathcal{R}$ are placed in such a way that they are at equal y-distance from the points of ∂D_{k-1}^δ with minimal or maximal y-coordinate, respectively.
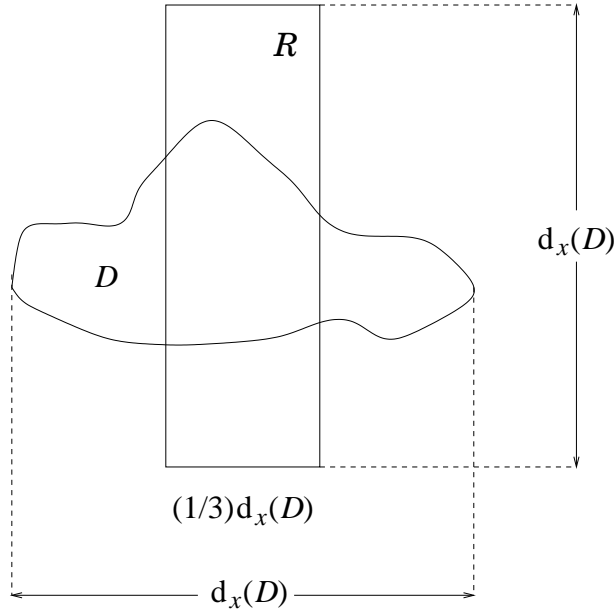

Figure 11: Schematic drawing of a domain D with $d_x(D) \geq d_y(D)$ and the associated rectangle $\mathcal{R}$.

It follows from the Russo-Seymour-Welsh lemma [31, 34] (see also [16, 20]) that the probability to have two vertical $\mathcal{T}$-crossings of $\mathcal{R}$ of different colors is bounded away from zero by a positive constant p_0 that does not depend on δ (for δ small enough). If that happens, then $\max_j d_x(D_{k,j}^\delta) \leq \frac{2}{3} d_x(D_{k-1}^\delta)$. The same argument of course applies to the maximal y-distance when $d_y(D_{k-1}^\delta) \geq d_x(D_{k-1}^\delta)$. We can summarize the above observation in the following lemma.

Lemma 6.4. *Suppose that at step k of the full discrete construction an exploration process γ_k^δ is run inside a domain D_{k-1}^δ. If $d_x(D_{k-1}^\delta) \geq d_y(D_{k-1}^\delta)$, then for δ small enough (i.e., $\delta \leq C\, d_x(D_{k-1}^\delta)$ for some constant C), $\max_j d_x(D_{k,j}^\delta) \leq \frac{2}{3} d_x(D_{k-1}^\delta)$ with probability at least p_0 independent of δ. The same holds for the maximal y-distances when $d_y(D_{k-1}^\delta) \geq d_x(D_{k-1}^\delta)$.*

Here is another lemma that will be useful later on. (For an example of the phenomenon described in the lemma, see Figure 6, and assume that the unexplored hexagons there are all blue; then the s-boundary of the small domain made of a single blue hexagon and that of the blue domain to the northeast share exactly two adjacent yellow hexagons.)

Lemma 6.5. *Two "daughter" subdomains, $D_{k,j}^\delta$ and $D_{k,j'}^\delta$, either have disjoint s-boundaries, or else their common s-boundary consists of exactly two adjacent hexagons (of the same color) where the exploration path γ_k^δ came within 2 hexagons of touching itself just when completing the s-boundary of one of the two subdomains.*

Proof. Suppose that the two daughter subdomains have s-boundaries $\Delta D_{k,j}^\delta$ and $\Delta D_{k,j'}^\delta$ that are not disjoint and let $S = \{\xi_1, \ldots, \xi_i\}$ be the set of (sites of $\mathcal{T}$ that are the centers of the) hexagons that belong to both s-boundaries. S can be partitioned into subsets consisting of single hexagons that are not adjacent to any another hexagon in S and groups of hexagons that form simple $\mathcal{T}$-paths (because the s-boundaries of the two subdomains are simple $\mathcal{T}$-loops). Let $\{\xi_l, \ldots, \xi_m\}$ be such a subset of hexagons of S that form a simple $\mathcal{T}$-path $\pi_0 = (\xi_l, \ldots, \xi_m)$. Then there is a $\mathcal{T}$-path π_1 of hexagons in $\Delta D_{k,j}^\delta$ that goes from ξ_l to ξ_m without using any other hexagon of π_0 and a different $\mathcal{T}$-path π_2 in $\Delta D_{k,j'}^\delta$ that goes from ξ_m to ξ_l without using any other hexagon of π_0. But then, all the hexagons in π_0 other than ξ_l and ξ_m are "surrounded" by $\pi_1 \cup \pi_2$ and therefore cannot have been explored by the exploration process that produced $D_{k,j}^\delta$ and $D_{k,j'}^\delta$, and cannot belong to $\Delta D_{k,j}^\delta$ or $\Delta D_{k,j'}^\delta$, leading to a contradiction, unless $\pi_0 = (\xi_l, \xi_m)$. Similar arguments lead to a contradiction if S is partitioned into more than one subset.

If $\xi_i \in S$ is not adjacent to any other hexagon in S, then it is adjacent to two other hexagons of $\Delta D_{k,j}^\delta$ and two hexagons of $\Delta D_{k,j'}^\delta$. Since ξ_i has only six neighbors and neither the two hexagons of $\Delta D_{k,j}^\delta$ adjacent to ξ_i nor those of $\Delta D_{k,j'}^\delta$ can be adjacent to each other, each hexagon of $\Delta D_{k,j}^\delta$ is adjacent to one of $\Delta D_{k,j'}^\delta$. But then, as before, ξ_i is "surrounded" by $\{\Delta D_{k,j}^\delta \cup \Delta D_{k,j'}^\delta\} \setminus \xi_i$ and therefore cannot have been explored by the exploration process that produced $D_{k,j}^\delta$ and $D_{k,j'}^\delta$, and cannot belong to $\Delta D_{k,j}^\delta$ or $\Delta D_{k,j'}^\delta$, leading once again to a contradiction. The proof is now complete, since the only case remaining is the one where S consists of a single pair of adjacent hexagons as stated in the lemma. $\square$

With these lemmas, we can now proceed with the proof of the second part of the theorem. Lemma 6.4 tells us that large domains are "chopped" with bounded away from zero probability ($\geq p_0 > 0$), but we need to keep track of domains of diameter larger than ε in such a way as to avoid "double counting" as the lattice construction proceeds. More accurately, we will keep track of domains $\tilde{D}^\delta$ having $\mathrm{d}_m(\tilde{D}^\delta) \geq \frac{1}{\sqrt{2}}\varepsilon$, since only these can have diameter larger than ε. To do so, we will associate with each domain $\tilde{D}^\delta$ having $\mathrm{d}_m(\tilde{D}^\delta) \geq \frac{1}{\sqrt{2}}\varepsilon$ that we encounter as we do the lattice construction a non-negative integer label. The first domain is $D_0^\delta = \mathbb{D}^\delta$ (see the beginning of Section 4.3) and this gets label 1. After each exploration process in a domain $\tilde{D}^\delta$ with $\mathrm{d}_m(\tilde{D}^\delta) \geq \frac{1}{\sqrt{2}}\varepsilon$, if the number $\tilde{m}$ of "daughter" subdomains $\tilde{D}_j^\delta$ with $\mathrm{d}_m(\tilde{D}_j^\delta) \geq \frac{1}{\sqrt{2}}\varepsilon$ is 0, then the label of $\tilde{D}^\delta$ is no longer

used, if instead $\tilde{m} \geq 1$, then one of these $\tilde{m}$ subdomains (chosen by any procedure – e.g., the one with the highest priority for further exploration) is assigned the *same* label as $\tilde{D}^\delta$ and the rest are assigned the next $\tilde{m} - 1$ integers that have never before been used as labels. Note that once all domains have $d_m < \frac{1}{\sqrt{2}}\varepsilon$, there are no more labelled domains.

Lemma 6.6. *Let M_ε^δ denote the total number of labels used in the above procedure; then for any fixed $\varepsilon > 0$, M_ε^δ is bounded in probability as $\delta \to 0$; i.e., $\lim_{M \to \infty} \limsup_{\delta \to 0} \mathbb{P}(M_\varepsilon^\delta > M) = 0$.*

Proof. Except for D_0^δ, every domain comes with (at least) a "physically correct" monochromatic "half-boundary" (notice that we are considering s-boundaries and that a half-boundary coming from the "artificially colored" boundary of D_0^δ is not considered a physically correct monochromatic half-boundary). Let us assume, without loss of generality, that $M_\varepsilon^\delta > 1$. If we associate with each label the "last" (in terms of steps of the discrete construction) domain which used that label (its daughter subdomains all had $d_m < \frac{1}{\sqrt{2}}\varepsilon$), then we claim that it follows from Lemma 6.5 that (with high probability) any two such last domains that are labelled have disjoint s-boundaries. This is a consequence of the fact that the two domains are subdomains of two "ancestors" that are distinct daughter subdomains of the *same* domain (possibly D_0^δ) and whose s-boundaries are therefore (by Lemma 6.5) either disjoint or else overlap at a pair of hexagons where an exploration path had a close encounter of distance two hexagons with itself. But since we are dealing only with macroscopic domains (of diameter at least order ε), such a close encounter would imply, like in Lemmas 5.1 and 5.2, the existence of six crossings, not all of the same color, of an annulus whose outer radius can be kept fixed while the inner radius is sent to zero together with δ. The probability of such an event goes to zero as $\delta \to 0$ and hence the unit disc $\mathbb{D}$ contains, with high probability, at least M_ε^δ disjoint monochromatic $\mathcal{T}$-paths of diameter at least $\frac{1}{\sqrt{2}}\varepsilon$, corresponding to the physically correct half-boundaries of the M_ε^δ labelled domains.

Now take the collection of squares s_j of side length $\varepsilon' > 0$ centered at the sites c_j of a scaled square lattice $\varepsilon'\mathbb{Z}^2$ of mesh size ε', and let $N(\varepsilon')$ be the number of squares of side ε' needed to cover the unit disc. Let $\varepsilon' < \varepsilon/2$ and consider the event $\{M_\varepsilon^\delta \geq 6\,N(\varepsilon')\}$, which implies that, with high probability, the unit disc contains at least $6\,N(\varepsilon')$ disjoint monochromatic $\mathcal{T}$-paths of diameter at least $\frac{1}{\sqrt{2}}\varepsilon$ and that, for at least one $j = j_0$, the square s_{j_0} intersects at least six disjoint monochromatic $\mathcal{T}$-paths of diameter larger that $\frac{1}{\sqrt{2}}\varepsilon$, so that the "annulus" $B(c_{j_0}, \frac{1}{2\sqrt{2}}\varepsilon) \setminus s_{j_0}$ is crossed by at least six disjoint monochromatic $\mathcal{T}$-paths contained inside the unit disc.

If all these $\mathcal{T}$-paths crossing $B(c_{j_0}, \frac{1}{2\sqrt{2}}\varepsilon) \setminus s_{j_0}$ have the same color, say blue, then since they are portions of boundaries of domains discovered by exploration processes, they are "shadowed" by exploration paths and therefore between at least one pair of blue $\mathcal{T}$-paths, there is at least one yellow $\mathcal{T}$-path crossing $B(c_{j_0}, \frac{1}{2\sqrt{2}}\varepsilon) \setminus s_{j_0}$. Therefore, whether the original monochromatic $\mathcal{T}$-paths are all of the same color or not, $B(c_{j_0}, \frac{1}{2\sqrt{2}}\varepsilon) \setminus s_{j_0}$ is crossed by at least six disjoint monochromatic $\mathcal{T}$-paths not all of the same color contained in the unit disc. Let $g(\varepsilon, \varepsilon')$ denote the $\limsup$ as $\delta \to 0$ of the probability that such an event

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happens anywhere inside the unit disc. We have shown that the event $\{M_\varepsilon^\delta \geq 6\, N(\varepsilon')\}$ implies a "six-arms" event unless not all labelled domains have disjoint s-boundaries. But the latter also implies a "six-arms" event, as discussed before; therefore

$$\limsup_{\delta \to 0} \mathbb{P}(M_\varepsilon^\delta \geq 6\, N(\varepsilon')) \leq 2\, g(\varepsilon, \varepsilon'). \tag{15}$$

Since $B(c_{j_0}, \frac{1}{2\sqrt{2}}\varepsilon) \setminus B(c_{j_0}, \frac{1}{\sqrt{2}}\varepsilon') \subset B(c_{j_0}, \frac{1}{2\sqrt{2}}\varepsilon) \setminus s_{j_0}$, bounds in [21] imply that, for ε fixed, $g(\varepsilon, \varepsilon') \to 0$ as $\varepsilon' \to 0$, which shows that

$$\lim_{M \to \infty} \limsup_{\delta \to 0} \mathbb{P}(M_\varepsilon^\delta > M) = 0 \tag{16}$$

and concludes the proof of the lemma. $\square$

Now, let N_i^δ denote the number of distinct domains that had label i (this is equal to the number of steps that label i survived). Let us also define $H(\varepsilon)$ to be the smallest integer $h \geq 1$ such that $(\frac{2}{3})^h < \frac{1}{\sqrt{2}}\varepsilon$ and G_h to be the random variable corresponding to how many Bernoulli trials (with probability p_0 of success) it takes to have h successes. Then, we may apply (sequentially) Lemma 6.4 to conclude that for any i

$$\mathbb{P}(N_i^\delta \geq k+1) \leq \mathbb{P}(G_{H(\varepsilon)} + G'_{H(\varepsilon)} \geq k), \tag{17}$$

where G'_h is an independent copy of G_h.

Now let $\tilde{N}_1(\varepsilon), \tilde{N}_2(\varepsilon), \ldots$ be i.i.d. random variables equidistributed with $G_{H(\varepsilon)} + G'_{H(\varepsilon)}$. Let $\tilde{K}_\delta(\varepsilon)$ be the number of steps needed so that all domains left to explore have $\mathrm{d}_m < \frac{1}{\sqrt{2}}\varepsilon$. Then, for any positive integer M,

$$\mathbb{P}(\tilde{K}_\delta(\varepsilon) > C) \leq \mathbb{P}(M_\varepsilon^\delta \geq M+1) + \mathbb{P}(\tilde{N}_1(\varepsilon) + \ldots + \tilde{N}_M(\varepsilon) \geq C). \tag{18}$$

Notice that, for fixed M, $\mathbb{P}(\tilde{N}_1(\varepsilon) + \ldots + \tilde{N}_M(\varepsilon) \geq C) \to 0$ as $C \to \infty$. Moreover, for any $\hat{\varepsilon} > 0$, by Lemma 6.6, we can choose $M_0 = M_0(\hat{\varepsilon})$ large enough so that $\limsup_{\delta \to 0} \mathbb{P}(M_\varepsilon^\delta > M_0) < \hat{\varepsilon}$. So, for any $\hat{\varepsilon} > 0$, it follows that

$$\limsup_{C \to \infty} \limsup_{\delta \to 0} \mathbb{P}(\tilde{K}_\delta(\varepsilon) > C) < \hat{\varepsilon}, \tag{19}$$

which implies that

$$\lim_{C \to \infty} \limsup_{\delta \to 0} \mathbb{P}(\tilde{K}_\delta(\varepsilon) > C) = 0. \tag{20}$$

To conclude this part of the proof, notice that the discrete construction cannot "skip" a contour and move on to explore its interior, so that all the contours with diameter larger than ε must have been found by step k if all the domains present at that step have diameter smaller than ε. Therefore, $K_\delta(\varepsilon) \leq \tilde{K}_\delta(\varepsilon)$, which shows that $K_\delta(\varepsilon)$ is bounded in probability as $\delta \to 0$.

For the first part of the theorem, we need to prove, for any fixed $k \in \mathbb{N}$, joint convergence in distribution of the first k steps of a suitably reorganized discrete construction to the first k steps of the continuum one. Later we will explain why this reorganized construction has the same scaling limit as the one defined in Section 4.3. For each k, the first k steps of the reorganized discrete construction will be coupled to the first k steps of the continuum one with suitable couplings in order to obtain the convergence in distribution of those steps of the discrete construction to the analogous steps of the continuum one; the proof will proceed by induction in k. We will explain how to reorganize the discrete construction as we go along; in order to explain the idea of the proof, we will consider first the cases $k = 1$, 2 and 3, and then extend to all $k > 3$.

$k = 1$. The first step of the continuum construction consists of an SLE_6 γ_1 from $-i$ to i inside $\mathbb{D}$. Correspondingly, the first step of the discrete construction consists of an exploration path γ_1^δ inside $\mathbb{D}^\delta$ from the e-vertex closest to $-i$ to the e-vertex closest to i. The convergence in distribution of γ_1^δ to γ_1 is covered by statement (S).

$k = 2$. The convergence in distribution of the percolation exploration path to chordal SLE_6 implies that we can couple γ_1^δ and γ_1 generating them as random variables on some probability space $(\Omega', \mathcal{B}', \mathbb{P}')$ such that $d(\gamma_1(\omega'), \gamma_1^\delta(\omega')) \to 0$ for all ω' as $k \to \infty$ (see, for example, Corollary 1 of [9]).

Now, let D_1 be the domain generated by γ_1 that is chosen for the second step of the continuum construction, and let $c_1 \in \mathcal{P}$ be the highest ranking point of $\mathcal{P}$ contained in D_1. For δ small enough, c_1 is also contained in $\mathbb{D}^\delta$; let $D_1^\delta = D_1^\delta(c_1)$ be the unique connected component of the set $\mathbb{D}^\delta \setminus \Gamma(\gamma_1^\delta)$ containing c_1 (this is well-defined with probability close to 1 for small δ); D_1^δ is the domain where the second exploration process is to be carried out. ¿From the proof of Lemma 5.2, we know that the boundaries ∂D_1^δ and ∂D_1 of the domains D_1^δ and D_1 produced respectively by the path γ_1^δ and γ_1 are close with probability close to one for δ small enough.

For the next step of the discrete construction, we choose the two e-vertices x_1 and y_1 in ∂D_1^δ that are closest to the points a_1 and b_1 of ∂D_1 selected for the coupled continuum construction (if the choice is not unique, we can select the e-vertices with any rule to break the tie) and call γ_2^δ the percolation exploration path inside D_1^δ from x_1 to y_1. It follows from [3] that $\{\gamma_1^\delta, \partial D_1^\delta, \gamma_2^\delta\}$ converge jointly in distribution along some subsequence to some limit $\{\tilde{\gamma}_1, \partial \tilde{D}_1, \tilde{\gamma}_2\}$. We already know that $\tilde{\gamma}_1$ is distributed like γ_1 and we can deduce from the joint convergence in distribution of $(\gamma_1^\delta, \partial D_1^\delta)$ to $(\gamma_1, \partial D_1)$ (Lemma 5.2), that $\partial \tilde{D}_1$ is distributed like ∂D_1. Therefore, if we call γ_2 the SLE_6 path inside D_1 from a_1 to b_1, Lemma 5.3 implies that $\tilde{\gamma}_2$ is distributed like γ_2 and indeed that, as $\delta \to 0$, $\{\gamma_1^\delta, \partial D_1^\delta, \gamma_2^\delta\}$ converge jointly in distribution to $\{\gamma_1, \partial D_1, \gamma_2\}$.

$k = 3$. So far, we have proved the convergence in distribution of the (paths and boundaries produced in the) first two steps of the discrete construction to the (paths and boundaries produced in the) first two steps of the discrete construction. The third step of the continuum construction consists of an SLE_6 path γ_3 from $a_2 \in \partial D_2$ to $b_2 \in \partial D_2$, inside the

domain D_2 with highest priority after the second step has been completed. Let $c_2 \in \mathcal{P}$ be the highest ranking point of $\mathcal{P}$ contained in D_2, D_2^δ the domain of the discrete construction containing c_2 after the second step of the discrete construction has been completed (this is well defined with probability close to 1 for small δ), and choose the two e-vertices x_2 and y_2 in ∂D_2^δ that are closest to the points a_2 and b_2 of ∂D_2 selected for the coupled continuum construction (if the choice is not unique, we can select the e-vertices with any rule to break the tie). The third step of the discrete construction consists of an exploration path γ_3^δ from x_2 to y_2 inside D_2^δ.

It follows from [3] that $\{\gamma_1^\delta, \partial D_1^\delta, \gamma_2^\delta, \partial D_2^\delta, \gamma_3^\delta\}$ converge jointly in distribution along some subsequence to some limit $\{\tilde{\gamma}_1, \partial \tilde{D}_1, \tilde{\gamma}_2, \partial \tilde{D}_2, \tilde{\gamma}_3\}$. We already know that $\tilde{\gamma}_1$ is distributed like γ_1, $\partial \tilde{D}_1$ like ∂D_1 and $\tilde{\gamma}_2$ like γ_2, and we would like to apply Lemma 5.3 to conclude that $\tilde{\gamma}_3$ is distributed like γ_3 and indeed that, as $\delta \to 0$, $(\gamma_1^\delta, \partial D_1^\delta, \gamma_2^\delta, \partial D_2^\delta, \gamma_3^\delta)$ converges in distribution to $(\gamma_1, \partial D_1, \gamma_2, \partial D_2, \gamma_3)$. In order to do so, we have to first show that $\partial \tilde{D}_2$ is distributed like ∂D_2. If D_2^δ is a subset of $\mathbb{D}^\delta \setminus \Gamma(\gamma_1^\delta)$, this follows from Lemma 5.2, as in the previous case, but if the s-boundary of D_2^δ contains hexagons of $\Gamma(\gamma_2^\delta)$, then we cannot use Lemma 5.2 directly, although the proof of the lemma can be easily adapted to the present case, as we now explain.

Indeed, the only difference is in the proof of claim (C) and is due to the fact that, when dealing with a domain of type 1 or 2, we cannot use the bound on the probability of three disjoint crossings of a semi-annulus because the domains we are dealing with may not be convex (like the unit disc). On the other hand, the discrete domains like D_1^δ and D_2^δ where we have to run exploration processes at various steps of the discrete construction are themselves generated by previous exploration processes, so that any hexagon of the s-boundary of such a domain has three adjacent hexagons which are the starting points of three disjoint $\mathcal{T}$-paths (two of one color and one of the other). Two of these $\mathcal{T}$-paths belong to the s-boundary of the domain, while the third belongs to the adjacent percolation cluster (see Figure 12). This allows us to use the bound on the probability of *six* disjoint crossings of an annulus.

To see this, let π_1, π_2 be the $\mathcal{T}$-paths contained in the s-boundary of the discrete domain (i.e., D_1^δ in the present context) and π_3 the $\mathcal{T}$-path belonging to the adjacent cluster, all starting from hexagons adjacent to some hexagon ξ (centered at u) in the s-boundary of D_1^δ. For $0 < \varepsilon' < \varepsilon$ and δ small enough, let $\mathcal{A}_u(\varepsilon, \varepsilon')$ be the event that the exploration path γ_2^δ enters the ball $B(u, \varepsilon')$ without touching ∂D_1^δ inside the larger ball $B(u, \varepsilon)$. $\mathcal{A}_u(\varepsilon, \varepsilon')$ implies having (at least) three disjoint $\mathcal{T}$-paths (two of one color and one of the other), π_4, π_5 and π_6, contained in D_1^δ and crossing the annulus $B(u, \varepsilon) \setminus B(u, \varepsilon')$, with π_4, π_5 and π_6 disjoint from π_1, π_2 and π_3. Hence, $\mathcal{A}_u(\varepsilon, \varepsilon')$ implies the event that there are (at least) six disjoint crossings (not all of the same color) of the annulus $B(u, \varepsilon) \setminus B(u, \varepsilon')$.

Once claim (C) is proved, the rest of the proof of Lemma 5.2 applies to the present case. Therefore, we have convergence in distribution of ∂D_2^δ to ∂D_2, which allows us to use Lemma 5.3 and conclude that $(\gamma_1^\delta, \gamma_2^\delta, \gamma_3^\delta)$ converges in distribution to $(\gamma_1, \gamma_2, \gamma_3)$.

$k > 3$. We proceed by induction in k, iterating the steps explained above; there are no new difficulties; all steps for $k \geq 4$ are analogous to the case $k = 3$.
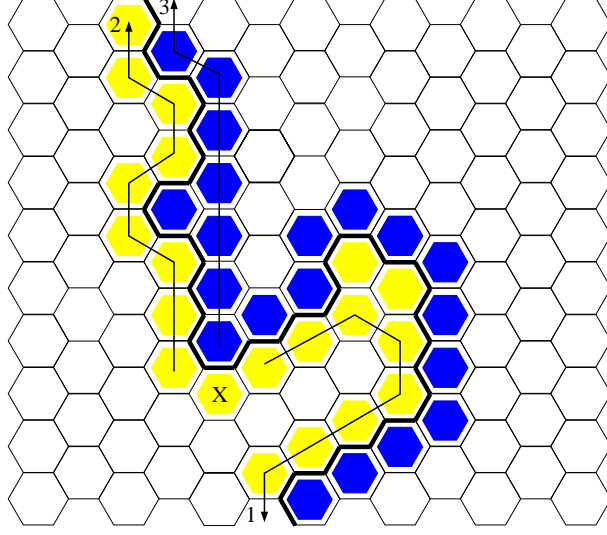
Figure 12: Hexagon X, in the s-boundary of the domain D_j^δ to the left of the exploration path indicated by a heavy line, has three neighbors that are the starting points of two disjoint yellow $\mathcal{T}$-paths (denoted 1 and 2) belonging to the s-boundary of D_j^δ and one blue $\mathcal{T}$-path (denoted 3) belonging to the adjacent percolation cluster.

To conclude the proof of the theorem, we need to show that the scaling limit of the original full discrete construction defined in Section 4.3 is the same as that of the reorganized one just used in the proof of the first part of the theorem. In order to do so, we can couple the two constructions by using the same percolation configuration for both, so that the two constructions have at their disposal the same set of loops to discover. We proved above that the original discrete construction finds *all* the "macroscopic" loops, so we have to show that this is true also for the reorganized version of the discrete construction. This is what we will do next, using essentially the same arguments as those employed for the original discrete construction; we present these arguments for the sake of completeness since there are some changes.

Consider the reorganized discrete construction described above, where the starting and ending points of the exploration processes at each step are chosen to be close to those of the corresponding (coupled) continuum construction. Suppose that at step k of this discrete construction an exploration process γ_k^δ is run inside a domain D_{k-1}^δ, and write $D_{k-1}^\delta \setminus \Gamma(\gamma_k^\delta) = \bigcup_j D_{k,j}^\delta$, where $\{D_{k,j}^\delta\}$ are the connected domains into which D_{k-1}^δ is split by the set $\Gamma(\gamma_k^\delta)$ of hexagons explored by γ_k^δ.

Let $d_x(D_{k-1})$ (resp., $d_x(D_{k-1}^\delta)$) and $d_y(D_{k-1})$ (resp., $d_y(D_{k-1}^\delta)$) be respectively the maximal x- and y-distance between pairs of points in ∂D_{k-1} (resp., ∂D_{k-1}^δ). If $d_x(D_{k-1}^\delta) \geq d_y(D_{k-1}^\delta)$ and the e-vertices on ∂D_{k-1}^δ are chosen to be closest to two points of ∂D_{k-1} with maximal x-distance, then the same construction and argument spelled out earlier in the first part of the proof (corresponding to the second part of the theorem) show that

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$\max_j \mathrm{d}_x(D^\delta_{k,j}) \leq \frac{2}{3}\mathrm{d}_x(D^\delta_{k-1})$ with bounded away from zero probability.

If the e-vertices on ∂D^δ_{k-1} are chosen to be closest to two points of ∂D_{k-1} with maximal x-distance but $\mathrm{d}_x(D^\delta_{k-1}) \leq \mathrm{d}_y(D^\delta_{k-1})$, then consider the rectangle $\mathcal{R}'$ whose vertical sides are aligned to the y-axis, have length $\mathrm{d}_y(D^\delta_{k-1})$, and are each placed at the same x-distance from the points of ∂D^δ_{k-1} with minimal or maximal x-coordinate in such a way that the horizontal sides of $\mathcal{R}'$ have length $\frac{1}{3}\mathrm{d}_y(D^\delta_{k-1})$; the bottom and top sides of $\mathcal{R}'$ are placed in such a way that they touch the points of ∂D^δ_{k-1} with minimal or maximal y-coordinate, respectively. Notice that, because of the coupling between the continuum and discrete constructions, for any $\tilde{\varepsilon} > 0$, for k large enough, $|\mathrm{d}_x(D^\delta_{k-1}) - \mathrm{d}_x(D_{k-1})| \leq \tilde{\varepsilon}$ and $|\mathrm{d}_y(D^\delta_{k-1}) - \mathrm{d}_y(D_{k-1})| \leq \tilde{\varepsilon}$. Since in the case under consideration we have $\mathrm{d}_y(D^\delta_{k-1}) \geq \mathrm{d}_x(D^\delta_{k-1})$ and $\mathrm{d}_x(D_{k-1}) \geq \mathrm{d}_y(D_{k-1})$, for δ large enough, we must also have $|\mathrm{d}_y(D^\delta_{k-1}) - \mathrm{d}_x(D^\delta_{k-1})| \leq 2\tilde{\varepsilon}$. Once again, it follows from the Russo-Seymour-Welsh lemma that the probability to have two vertical $\mathcal{T}$-crossings of $\mathcal{R}'$ of different colors is bounded away from zero by a positive constant that does not depend on δ (for δ small enough). If that happens, then $\max_j \mathrm{d}_x(D^\delta_{k,j}) \leq \frac{2}{3}\mathrm{d}_x(D^\delta_{k-1}) + \frac{1}{3}\tilde{\varepsilon}$.

All other cases are handled in the same way, implying that the maximal x- and y-distances of domains that appear in the discrete construction have a positive probability (bounded away from zero) to decrease by (approximately) a factor $2/3$ at each step of the discrete construction in which an exploration process is run in that domain.

With this result at our disposal, the rest of the proof, that for any $\varepsilon > 0$ the number of steps needed to find all the loops of diameter larger than ε is bounded in probability as $\delta \to 0$ (which implies that *all* the "macroscopic" loops are discovered), proceeds exactly like for the original discrete construction. $\square$

Proof of Theorem 6. First of all, we want to show that $P^D \equiv \hat{I}_D P_R$ does not depend on R, provided D is strictly contained in $\mathbb{D}_R$ and $\partial D \cap \partial \mathbb{D}_R = \emptyset$. In order to do this, we assume that the above conditions are satisfied for the pair D, R and show that $\hat{I}_D P_R = \hat{I}_D P_{R'}$ for all $R' > R$.

Take two copies of the scaled hexagonal lattice, $\delta\mathcal{H}$ and $\delta\mathcal{H}'$, their dual lattices $\delta\mathcal{T}$ and $\delta\mathcal{T}'$, and two percolation configurations, $\sigma_{\mathbb{D}_R}$ and $\sigma'_{\mathbb{D}_{R'}}$, both with blue boundary conditions and coupled in such a way that $\sigma_{\mathbb{D}_R} = \sigma'_{\mathbb{D}_R}$. The laws of the boundaries of σ and σ' are also coupled, in such a way that the boundaries or portions of boundaries contained inside D are identical for all small enough δ. Therefore, letting $\delta \to 0$ and using the convergence of the percolation boundaries inside $\mathbb{D}_R$ and $\mathbb{D}_{R'}$ to the continuum nonsimple loop processes P_R and $P_{R'}$ respectively, we conclude that $\hat{I}_D P_R = \hat{I}_D P_{R'}$.

¿From what we have just proved, it follows that the probability measures $P^{\mathbb{D}_R}$ on $(\Omega_R, \mathcal{B}_R)$, for $R \in \mathbb{R}_+$, satisfy the consistency conditions $P^{\mathbb{D}_{R_1}} = \hat{I}_{\mathbb{D}_{R_1}} P^{\mathbb{D}_{R_2}}$ for all $R_1 \leq R_2$. Since Ω_R, Ω are complete separable metric spaces, the measurable spaces $(\Omega_R, \mathcal{B}_R)$, $(\Omega, \mathcal{B})$ are standard Borel spaces and so we can apply Kolmogorov's extension theorem (see, for example, [15]) and conclude that there exists a unique probability measure on $(\Omega, \mathcal{B})$ with $P^{\mathbb{D}_R} = \hat{I}_{\mathbb{D}_R} P$ for all $R \in \mathbb{R}_+$. It follows that, for $R' > R$ and all D strictly contained in $\mathbb{D}_R$ and such that $\partial D \cap \partial \mathbb{D}_R = \emptyset$, $\hat{I}_D P_R = P^D = \hat{I}_D P_{R'} = \hat{I}_D \hat{I}_{\mathbb{D}_R} P_{R'} = \hat{I}_D P^{\mathbb{D}_R} = \hat{I}_D \hat{I}_{\mathbb{D}_R} P = \hat{I}_D P$, which concludes the proof. $\square$

Proof of Theorems 1 and 3. These are immediate consequences of Theorems 5 and 6, where the full scaling limit is intended in the topology induced by (5). □

Proof of Theorem 2. *1.* The fact that the Continuum Nonsimple Loop process is a random collection of noncrossing continuous loops is a direct consequence of its definition. The fact that the loops touch themselves is a consequence of their being constructed out of SLE_6, while the fact that they touch each other follows from the observation that a chordal SLE_6 path $\gamma_{D,a,b}$ touches ∂D with probability one. Therefore, each new loop in the continuum construction touches one or more previous ones (many times).

The nonexistence of triple points follows directly from Lemma 5 of [21] on the number of crossings of an annulus, combined with Theorem 1, which allows to transport discrete results to the continuum case. In fact, a triple point would imply, for discrete percolation, at least six crossings (not all of the same color) of an annulus whose ratio of inner to outer radius goes to zero in the scaling limit, leading to a contradiction.

2. This follows from straightforward Russo-Seymour-Welsh type arguments for percolation (for more details, see, for example, Lemma 3 of [21]), combined with Theorem 1.

3. Combining Russo-Seymour-Welsh type arguments for percolation (see, for example, Lemma 3 of [21]) with Theorem 1, we know that P-a.s. there exists a (random) $R^* = R^*(R)$, with $R^* < \infty$, such that $\mathbb{D}_R$ is surrounded by a continuum nonsimple loop contained in $\mathbb{D}_{R^*}$. ¿From (the proof of) Theorem 6, we also know that $\hat{I}_{\mathbb{D}_{R''}}P = P^{\mathbb{D}_{R''}} = I_{\mathbb{D}_{R''}}P_{R'}$ for all $R' > R''$. This implies that by taking R' large enough and performing the continuum construction inside $\mathbb{D}_{R'}$, we have a positive probability of generating a loop λ contained in the annulus $\mathbb{D}_{R''} \setminus \mathbb{D}_R$, with $R' > R'' > R$. If that is the case, all the loops contained inside $\mathbb{D}_R$ are connected, by construction, to the loop λ surrounding $\mathbb{D}_R$ by a finite sequence (a "path") of loops (remember that in the continuum construction each loop is generated by pasting together portions of SLE_6 paths inside domains whose boundaries are determined by previously formed loops or excursions). Therefore, any two loops contained inside $\mathbb{D}_R$ are connected to each other by a "path" of loops.

Using again the fact that $\hat{I}_{\mathbb{D}_{R''}}P = P^{\mathbb{D}_{R''}} = I_{\mathbb{D}_{R''}}P_{R'}$ for all $R' > R''$, and letting first R' and then R'' go to ∞, we see from the discussion above (with $R \to \infty$ as well) that any two loops are connected by a finite "path" of intermediate loops, P-a.s. □

Proof of Theorem 7. Combining Russo-Seymour-Welsh type arguments for percolation (see, for example, Lemma 3 of [21]) with Theorem 1, we know that P-a.s. there exists a bounded continuum nonsimple loop that surrounds both λ_1 and λ_2, so that $\tilde{D}_1$ and $\tilde{D}_2$ are both bounded. We can then take $R < \infty$ such that λ_1 and λ_2 (and therefore $\tilde{D}_1$ and $\tilde{D}_2$) are both contained in the disc $\mathbb{D}_R$ with probability tending to 1 as $R \to \infty$.

Consider now the continuum construction inside the disc $\mathbb{D}_R$ for some large R. Let λ'_1 (resp., λ'_2) be the smallest loop surrounding D_1 (resp., D_2) produced by the construction and let $\tilde{D}'_1$ (resp., $\tilde{D}'_2$) be the connected component of $\mathbb{R}^2 \setminus \lambda'_1$ (resp., $\mathbb{R}^2 \setminus \lambda'_2$) that contains D_1 (resp., D_2). It follows from the previous observation and from (the proof of) Theorem 6

that as $R \to \infty$, $\tilde{D}'_1$ (resp., $\tilde{D}'_2$) is (with probability tending to 1) distributed like $\tilde{D}_1$ (resp., $\tilde{D}_2$) and moreover the loop configuration inside $\tilde{D}'_1$ (resp., $\tilde{D}'_2$) is distributed by $P_{\tilde{D}_1}$ (resp., $P_{\tilde{D}_2}$).

This already proves the first claim of the theorem, since it is clear from the continuum construction inside $\mathbb{D}_R$ that the loop configurations inside $\tilde{D}'_1$ and $\tilde{D}'_2$ are independent. It also means that in order to complete the proof of theorem, it suffices to prove the second claim for the case of the continuum construction inside $\mathbb{D}_R$, for all large R. In order to do that, we consider a modified discrete construction inside $\mathbb{D}_R$, as explained below. In view of the above observations, we take R large and condition on the existence inside $\mathbb{D}_R$ of two disjoint loops, λ_1^δ and λ_2^δ, surrounding D_1 and D_2 respectively, and let $\tilde{D}_1^\delta$ (resp., $\tilde{D}_2^\delta$) be the domain of $\mathbb{D}_R^\delta \setminus \Gamma(\lambda_1^\delta)$ (resp., $\mathbb{D}_R^\delta \setminus \Gamma(\lambda_2^\delta)$) containing D_1 (resp., D_2).

The modified discrete construction inside $\mathbb{D}_R$ is analogous to the "ordinary" one except inside the domains $\tilde{D}_1^\delta$ and $\tilde{D}_2^\delta$, where the the exploration paths are coupled to a continuum construction inside the unit disc in the following way. Roughly speaking, the discrete construction inside $\tilde{D}_1^\delta$ is one in which the (x, y) pairs (the starting and ending points of the exploration paths) at each step are chosen to be closest to the $(\phi_\delta(a), \phi_\delta(b))$ points in $\tilde{D}_1^\delta$ mapped from the unit disc $\mathbb{D}$ via ϕ_δ, where the pairs (a, b) are those that appear at the corresponding steps of the continuum construction inside $\mathbb{D}$ and ϕ_δ is a certain conformal map from $\mathbb{D}$ onto $\tilde{D}_1^\delta$, as specified below. The discrete construction inside $\tilde{D}_2^\delta$ is coupled in the same way to the *same* continuum construction inside $\mathbb{D}$ via a certain conformal map ψ_δ from $\mathbb{D}$ onto $\tilde{D}_2^\delta$.

The conformal map ϕ_δ will be defined for δ sufficiently small and is specified in the following way. We fix a point z_0 in $\tilde{D}'_1$ and denote by ϕ the unique conformal map from $\mathbb{D}$ onto $\tilde{D}'_1$ such that $\phi(0) = z_0$ and $\phi'(0) > 0$. For δ sufficiently small, so that z_0 is contained in $\tilde{D}_1^\delta$, we let ϕ_δ be the unique conformal map from $\mathbb{D}$ onto $\tilde{D}_1^\delta$ such that $\phi_\delta(0) = z_0$ and $\phi'_\delta(0) > 0$.

We also denote by ψ the unique conformal map from $\mathbb{D}$ onto $\tilde{D}'_2$ such that $\psi(0) = g(z_0)$ and $\text{sign}(\psi'(0)) = \text{sign}(g'(z_0))$, where g is any fixed conformal map from $\tilde{D}'_1$ to $\tilde{D}'_2$. Note that, by the uniqueness part of Riemann's mapping theorem, we can conclude that $\psi = g \circ \phi$. For δ sufficiently small, so that $g(z_0)$ is contained in $\tilde{D}_2^\delta$, we let ψ_δ be the unique conformal map from $\mathbb{D}$ onto $\tilde{D}_2^\delta$ such that $\psi_\delta(0) = g(z_0)$ and $\text{sign}(\psi'_\delta(0)) = \text{sign}(g'(z_0))$.

As $\delta \to 0$, $\tilde{D}_1^\delta \to \tilde{D}'_1$ and $\tilde{D}_2^\delta \to \tilde{D}'_2$, and by an application of Radó's theorem (Theorem 4), (the continuous extensions of) ϕ_δ and ψ_δ converge uniformly in $\overline{\mathbb{D}}$ to the (continuous extensions of) ϕ and ψ respectively.

We now describe more precisely the modified construction inside $\tilde{D}_1^\delta$. Let γ_1 be the first SLE_6 path in $\mathbb{D}$ from a_1 to b_1; because of the conformal invariance of SLE_6, the image $\phi_\delta(\gamma_1)$ of γ_1 under ϕ_δ is a path distributed as the trace of chordal SLE_6 in $\tilde{D}_1^\delta$ from $\phi_\delta(a_1)$ to $\phi_\delta(b_1)$. The uniform convergence of ϕ_δ to ϕ and statement (S) imply that the exploration path γ_1^δ inside $\tilde{D}_1^\delta$ from x_1 to y_1, chosen to be closest to $\phi_\delta(a_1)$ and $\phi_\delta(b_1)$ respectively, converges in distribution to $\phi(\gamma_1)$, as $\delta \to 0$, which means that there exists a coupling so that the paths γ_1^δ and $\phi_\delta(\gamma_1)$ stay close for δ small.

One can use the same strategy as in the proof of the first part of Theorem 5, and obtain a discrete construction whose exploration paths are coupled to the SLE_6 paths

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$\phi_\delta(\gamma_k)$ that are the images of the paths γ_k in $\mathbb{D}$. Then, for this discrete construction inside $\tilde{D}_1^\delta$, the scaling limit of the exploration paths will be paths inside $\tilde{D}_1'$ distributed as the images of the SLE_6 paths in $\mathbb{D}$ under the conformal map ϕ.

Analogously, for the discrete construction inside $\tilde{D}_2^\delta$, the scaling limit of the exploration paths will be paths inside $\tilde{D}_2'$ distributed as the images of the SLE_6 paths in $\mathbb{D}$ under the (continuous extension of the) conformal map $\psi : \mathbb{D} \to \tilde{D}_2'$ which is the uniform limit of ψ_δ as $\delta \to 0$.

Therefore, the path inside $\tilde{D}_2'$ obtained as the scaling limit of an exploration path at a given step of the construction inside $\tilde{D}_2^\delta$ is the image under the conformal map $g = \psi \circ \phi^{-1} : \tilde{D}_1' \to \tilde{D}_2'$ of the path inside $\tilde{D}_1'$ obtained as the scaling limit of the exploration path at corresponding step of the construction inside $\tilde{D}_1^\delta$.

In order to conclude the proof, we have to show that the discrete constructions inside $\tilde{D}_1^\delta$ and $\tilde{D}_2^\delta$ defined above find all the boundaries (with diameter greater than ε) in a number of steps that is bounded in probability as $\delta \to 0$. This is as in the second part of Theorem 5, but since we are now dealing with a modified construction inside general Jordan domains, we need to show that we can reach the same conclusion. In order to do so, we will use the fact that the modified construction is coupled to an "ordinary" continuum construction in the unit disc. We work out the details only for $\tilde{D}_1^\delta$, since the proof is the same for $\tilde{D}_2^\delta$.

¿From the second part of Theorem 5 it follows that, for any fixed $\varepsilon > 0$ and $M < \infty$, the probability that the number of steps of the continuum construction in $\mathbb{D}$ that are necessary to ensure that only domains with diameter less than ε/M are present is larger than C, goes to zero as $C \to \infty$. Since ϕ_δ (can be extended to a function that) is continuous in the compact set $\overline{\mathbb{D}}$, ϕ_δ is uniformly continuous and so we can choose $M_\delta = M(\phi_\delta) < \infty$ such that any subdomain of $\mathbb{D}$ of diameter at most ε/M_δ is mapped by ϕ_δ to a subdomain of $\tilde{D}_1^\delta$ of diameter at most ε.

Since $\phi_\delta \to \phi$, as $\delta \to 0$, where ϕ (can be extended to a function that) is continuous in the compact set $\overline{\mathbb{D}}$ and is therefore uniformly continuous, we can choose $M_0 = M(\phi) < \infty$ such that any subdomain of $\mathbb{D}$ of diameter at most ε/M_0 is mapped by ϕ_δ to a subdomain of $\tilde{D}_1'$ of diameter at most ε, and moreover such that $\limsup_{\delta \to 0} M_\delta \le M_0$.

This, combined with the coupling between SLE_6 paths and exploration paths inside $\tilde{D}^\delta$, assures that the number of steps necessary for the new discrete construction inside $\tilde{D}_1^\delta$ to find all the loops of diameter at least ε is bounded in probability as $\delta \to 0$.

Therefore, the scaling limit, as $\delta \to 0$, of the modified discrete constructions for $\tilde{D}_1^\delta$ and $\tilde{D}_2^\delta$ give the measures $P_{\tilde{D}_1'}$ and $P_{\tilde{D}_2'}$, and it follows by construction that $P_{\tilde{D}_2'} = g * P_{\tilde{D}_1'}$ for *any* conformal map g from $\tilde{D}_1'$ onto $\tilde{D}_2'$. Since this is true for all large R, by letting $R \to \infty$ we can conclude that $P_{\tilde{D}_2} = f * P_{\tilde{D}_1}$. $\square$

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